Compositions of EGF

α = structure s.t. $\alpha(\emptyset) = \emptyset$

For finite set S, let $\Pi_S = \{$set partitions of $S\}$

Define $e^\alpha(S) = \coprod_{\{S_1,\dots,S_k\} \in \Pi_S} \alpha(S_1) \times \cdots \times \alpha(S_k)$

<u>Thm</u>. $E_{e^\alpha}(x) = e^{E_\alpha(x)} = \sum_{k\geq 0} \frac{E_\alpha(x)^k}{k!}$

<u>Pf</u>. Since $|\alpha(\emptyset)| = 0$, we have $[x^n]\, E_\alpha(x)^k = 0$ if $k > n$.

$$\Rightarrow [x^n]\, e^{E_\alpha(x)} = [x^n] \sum_{k\geq 0} \frac{E_\alpha(x)^k}{k!} = [x^n] \sum_{k=0}^{n} \frac{E_\alpha(x)^k}{k!}$$

$[x^n]\, E_\alpha(x)^k$ = # of ways to choose ordered set partition of $[n]$ into k blocks, put α-structure on each block.

Dividing by $k!$ removes order.

Summing over $k=0$ to n allows any # of blocks. $= [x^n]\, E_{e^\alpha}(x)$. □

<u>Ex</u>. $\alpha(S) = \begin{cases} \{1\} & \text{if } |S| > 0 \\ \emptyset & \text{if } |S| = 0 \end{cases}$.

Then $e^\alpha(S) = \Pi_S$, so $e^\alpha([n]) = B(n)$ (Bell number)

$$\Rightarrow \sum_{n\geq 0} B(n) \frac{x^n}{n!} = E_{e^\alpha}(x) = e^{E_\alpha(x)} = \exp(e^x - 1).$$

General heuristic: If $\alpha(S)$ consists of "connected" structures, then $e^\alpha(S)$ consists of arbitrary structures

Ex. An undirected graph is connected if $\forall\, x,y$ vertices, $\exists$ walk from x to y. In general, $x \sim y$ to mean $\exists$ walk from x to y. Equivalence classes are connected components of the graph.

For finite set S, let $\beta(S) = \{\text{simple graphs whose vertex set is } S\}$

$$|\beta(S)| = 2^{\binom{|S|}{2}}.$$

If $|S|>0$, let $\alpha(S) = \{\text{simple connected graphs w/ vertex set } S\}$
$$\alpha(\emptyset) = \emptyset$$

Note: $E_\beta(x) = e^{E_\alpha(x)}$ since $\beta = e^\alpha$.

$$\Rightarrow \quad E_\alpha(x) = \mathbb{L}(E_\beta(x)).$$

Ex. Every permutation has a cycle decomposition

Think of cycles as "connected" permutations

For $S \neq \emptyset$, let $\alpha(S) = \{\text{cyclic orderings of } S\}$
$$\alpha(\emptyset) = \emptyset.$$

$$|\alpha(S)| = (|S|-1)! \quad \text{if } |S|>0.$$

$e^\alpha(S) = \{\text{permutations of } S\}$, $\quad |e^\alpha(S)| = |S|!$

$$\exp\left(\sum_{n\geq 1} \frac{x^n}{n}\right) = e^{E_\alpha(x)} = E_{e^\alpha}(x) = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

$$\Rightarrow E_\alpha(x) = \mathbb{L}\left(\frac{1}{1-x}\right).$$

Note: $\mathbb{L}\left(\frac{1}{1-x}\right)$ is unique formal power series $F(x)$ s.t. $F(0)=0$ & $DF = \dfrac{D\left(\frac{1}{1-x}\right)}{(1-x)^{-1}} = \dfrac{1}{1-x}$

Ex. A bijection $f: [n] \to [n]$ is an involution if $f \circ f = id$.

This is a permutation s.t. all cycles are length 1 or 2.

"Connected" involution is cycle of length 1 or 2.

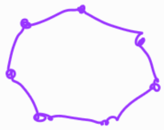
If $|S| \in \{1, 2\}$, let $\alpha(S) = \{$cyclic permutations of $S\}$

Else, $\alpha(S) = \emptyset$. $\quad E_\alpha(x) = x + \frac{x^2}{2}$

$$E_{e^\alpha}(x) = \exp(E_\alpha(x)) = \exp\left(x + \frac{x^2}{2}\right) = \sum_{n \geq 0} \#\text{involutions of } [n] \cdot \frac{x^n}{n!}$$

Ex. Consider simple graphs s.t. every vertex has degree 2

i.e., contained in exactly 2 edges.

If connected, graph looks like  (w/ at least 3 vertices)

If $|S| \geq 3$, let $\alpha(S) = \{$cycle graphs w/ vertex set $S\}$

otherwise, $\alpha(S) = \emptyset$

If $|S| \geq 3$, $\quad |\alpha(S)| = \frac{(|S|-1)!}{2}$.

$$E_{e^\alpha}(x) = \exp\left(\frac{1}{2} \sum_{n \geq 3} \frac{x^n}{n}\right) = \exp\left(\frac{1}{2} \mathbb{L}\left(\frac{1}{1-x}\right) - \frac{x}{2} - \frac{x^2}{4}\right)$$

$$= \frac{\exp\left(-\frac{x}{2} - \frac{x^2}{4}\right)}{\sqrt{1-x}}$$

α = structure s.t. $\alpha(\emptyset) = \emptyset$

β = structure.

Define $$(\beta \circ \alpha)(S) = \coprod_{\{S_1, \ldots, S_k\} \in \Pi_S} \beta([k]) \times \alpha(S_1) \times \cdots \times \alpha(S_k)$$

Thm $\quad E_{\beta \circ \alpha}(x) = E_\beta(E_\alpha(x))$.