Thm (Lagrange inversion formula). Let $G(x)$ be formal power series s.t. $G(0) \neq 0$. Let $A(x)$ formal power series s.t. $A(0)=0$, and

$$A(x) = x\, G(A(x)).$$

Then for k, n we have

$$n\,[x^n]\, A(x)^k = k\,[x^{n-k}]\left(G(x)^n\right).$$

Ex. $C(x) = \sum_{n \geq 0} C_n x^n$ (Catalan)

satisfies $C(x) = 1 + x\, C(x)^2$. Define $A(x) = C(x) - 1$

$$A(x) \cancel{+1} = \cancel{1} + x\,(A(x)+1)^2$$

$$A(x) = x\,(A(x)+1)^2, \qquad G(x) = (1+x)^2$$

$$n\,[x^n]\, A(x) = [x^{n-1}]\,(1+x)^{2n} = \binom{2n}{n-1}$$

If $n > 0$, $C_n = \frac{1}{n}\binom{2n}{n-1} = \frac{(2n)!}{n\cdot(n-1)!\,(n+1)!} = \frac{1}{n+1}\binom{2n}{n}$

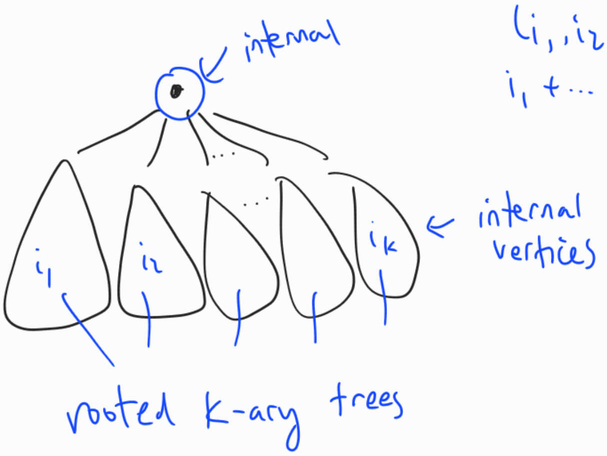
Ex. C_n = # rooted binary trees w/ $(n+1)$-leaves

= # rooted binary trees w/ n internal vertices.

Let c_n = # rooted k-ary trees w/ n internal vertices

Recurrence relation: $c_n = \sum_{\substack{(i_1, i_2, \dots, i_k) \\ i_1 + \dots + i_k = n-1}} c_{i_1} c_{i_2} \cdots c_{i_k}$ if $n > 0$

← all weak compositions of $n-1$ into k parts



Define $F(x) = \sum_{n \geq 0} c_n x^n$

$$F(x) = 1 + \sum_{n \geq 0} c_n x^n = 1 + x \sum_{n>0} \sum_{\substack{i_1, \dots, i_k \\ i_1 + \dots + i_k = n-1}} c_{i_1} \cdots c_{i_k} x^{n-1} = 1 + x F(x)^k$$

Define $A(x) = F(x) - 1$: $\quad A(x) + \cancel{1} = \cancel{1} + x(A(x)+1)^k$

$G(x) = (1+x)^k$

Lagrange: $\quad n[x^n] A(x) = [x^{n-1}](1+x)^{kn} = \binom{kn}{n-1}$

$$c_n = \frac{1}{n}\binom{kn}{n-1} = \frac{1}{(k-1)n+1}\binom{kn}{n}.$$

Proof of Lagrange inversion

<u>Lemma</u>. If $F(0) = 0$, then $\exists\, G(x)$ s.t. $G(F(x)) = x$

$\iff [x^1] F(x) \neq 0$.

Formal Laurent series: series of the form $\sum_{n \geq N} a_n x^n$ where N is some integer (depending on the series)

Only finitely many negative powers at a given time.

Addition / multiplication defined as before

Every nonzero formal Laurent series has a multiplicative inverse.

derivative: $\quad D \sum_{n \geq N} a_n x^n = \sum_{n \geq N} n a_n x^{n-1}$

Note $[x^{-1}] D F = 0$ $\forall$ formal Laurent series

Start w/ $A(x) = x\, G(A(x))$. $\quad (*)$

$[x^1] A(x) = [x^0] G(A(x)) = G(0) \neq 0$

$\Rightarrow \exists\, B(x)$ s.t. $B(A(x)) = x = A(B(x))$.

Substitute $x \to B(x)$ in $(*)$: $\quad x = B(x)\, G(x)$.

Write $A(x)^k = \sum_{d \geq k} \alpha_d x^d$. Substitute $x \to B(x)$ to get

$$x^k = \sum_{d \geq k} \alpha_d B(x)^d$$

Take derivative

$$k x^{k-1} = \sum_{d \geq k} \alpha_d \, d \cdot B(x)^{d-1} B'(x)$$

Divide by $B(x)^n$ (use formal Laurent series)

$$\frac{k x^{k-1}}{B(x)^n} = \sum_{d \geq k} d \alpha_d B(x)^{d-n-1} B'(x)$$

Take $[x^{-1}]$

If $d \neq n$, then $\underbrace{B(x)^{d-n-1} B'(x)}_{\text{no } [x^{-1}] \text{ term}} = \left(\frac{B(x)^{d-n}}{d-n} \right)'$

$$[x^{-1}] \frac{k x^{k-1}}{B(x)^n} = [x^{-1}] n \alpha_n \frac{B'(x)}{B(x)} \overset{\text{exercise}}{=} n \alpha_n = n [x^n] A(x)^k$$

$$\|$$

$$\Rightarrow [x^{-1}] \frac{k x^{k-1}}{B(x)^n} = [x^{-1}] k x^{k-n-1} G(x)^n = k [x^{k-n}] G(x)^n. \quad \square$$