$a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $t^2 - c_1 t - c_2 = (t-r_1)(t-r_2)$

$$\boxed{\begin{array}{l}\underline{\text{Thm.}}\ \text{If } r_1 = r_2, \text{ then } \exists \text{ constants } \alpha_1, \alpha_2 \text{ s.t.} \\ \qquad a_n = \alpha_1 r_1^n + n\alpha_2 r_1^n \quad \text{for all } n \geq 0\end{array}}$$

To solve for α_1, α_2, plug in $n=0,1$:

$n=0$: $a_0 = \alpha_1$

$n=1$: $a_1 = \alpha_1 r_1 + \alpha_2 r_1$

<u>Example:</u> $a_n = 4a_{n-1} - 4a_{n-2}$, $t^2 - 4t + 4 = (t-2)^2$

$a_0 = 1, a_1 = 1$

$n=0$: $1 = \alpha_1$, $n=1$: $1 = 2\alpha_1 + 2\alpha_2 \Rightarrow \alpha_2 = \frac{1-2}{2} = -\frac{1}{2}$

$$\Rightarrow a_n = 2^n + n\left(-\tfrac{1}{2}\right)2^n = 2^n\left(1 - \tfrac{n}{2}\right) = 2^{n-1}(2-n)$$

<u>Pf1</u> (formal power series) work as before, $A(x) = \sum_{n\geq 0} a_n x^n$

$$\cdots \quad A(x) = \frac{a_0 + (a_1 - c_1 a_0)x}{(1-r_1 x)^2}$$

$\exists$ constants β_1, β_2

$$= \frac{\beta_1}{1-r_1 x} + \frac{\beta_2}{(1-r_1 x)^2}$$

$$= \beta_1 \sum_{n\geq 0} (r_1 x)^n + \beta_2 \sum_{n\geq 0} (n+1)(r_1 x)^n$$

$$\frac{1}{(1-x)^2} = \left(\sum_n x^n\right)\left(\sum_n x^n\right)$$

$$(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$a_n = \beta_1 r_1^n + \beta_2 (n+1) r_1^n = (\beta_1 + \beta_2) r_1^n + \beta_2 n r_1^n$$

$$\alpha_1 = \beta_1 + \beta_2, \quad \alpha_2 = \beta_2. \qquad \square$$

<u>Pf2</u> (matrices) $C = \begin{bmatrix} c_1 & c_2 \\ 1 & 0 \end{bmatrix}$, char. poly is $(t-r_1)^2$ $\quad r = r_1$

~~not diagonalizable: if so,~~ $\cancel{C = B\begin{bmatrix} r_1 & 0 \\ 0 & r_1 \end{bmatrix}B^{-1} = \begin{bmatrix} r_1 & 0 \\ 0 & r_1 \end{bmatrix}}$

Jordan normal form: $\exists$ invertible 2×2 B s.t.

$$C = B\begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}B^{-1}.$$

Why is this possible? ① We know C has eigenvector v, $Cv = rv$.

② Choose basis $\{v, w\}$, then $Cw = av + bw$

In this basis $C \rightsquigarrow \begin{bmatrix} r & a \\ 0 & b \end{bmatrix}$ $\det C = r^2 = rb \Rightarrow b = r$

$a \neq 0$ since C not diagonalizable.

③ Use basis $\{v, w'\}$ where $w' = w/a$. Take $B = \begin{bmatrix} | & | \\ v & w' \\ | & | \end{bmatrix}$.

$C^n = B \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}^n B^{-1}$. By induction: $\begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}^n = \begin{bmatrix} r^n & nr^{n-1} \\ 0 & r^n \end{bmatrix}$

Finish as before: set $\begin{bmatrix} x \\ y \end{bmatrix} = B^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix}$.

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = C^n \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = B \begin{bmatrix} r & 1 \\ 0 & r \end{bmatrix}^n B^{-1} \begin{bmatrix} a_1 \\ a_0 \end{bmatrix} = B \begin{bmatrix} r^n & nr^{n-1} \\ 0 & r^n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= B \begin{bmatrix} r^n x + nr^{n-1} y \\ r^n y \end{bmatrix} \Longrightarrow$$ a_n is a linear combination of r^n and nr^{n-1}

(to match thm, note $\alpha_1 r^n + \alpha_2 n r^n = \alpha_1 r^n + \left(\frac{\alpha_2}{r}\right) n r^{n-1}$.) $\square$

Higher order relations

$a_n = c_1 a_{n-1} + \cdots + c_d a_{n-d}$ for $n \geq d$

① Factor $t^n - c_1 t^{n-1} - c_2 t^{n-2} - \cdots - c_d = (t - r_1)(t - r_2) \cdots (t - r_d)$

②(a) If all $r_1, \ldots, r_d$ distinct, then $\exists\ \alpha_1, \ldots, \alpha_d$ s.t.

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \cdots + \alpha_d r_d^n$$

③ Solve for $\alpha_1, \ldots, \alpha_d$ by plugging in $n = 0, 1, \ldots d-1$ to get d eqns w/ d variables.

(b) If not all distinct, if r_i appears w/ multiplicity m_i then we have solutions $(r_i^n), (nr_i^n), (n^2 r_i^n), \ldots, (n^{m_i - 1} r_i^n)$

Ex. If $d = 5$, roots are r_1, r_1, r_1, r_2, r_2 $(r_1 \neq r_2)$

then $a_n = \alpha_1 r_1^n + n\alpha_2 r_1^n + n^2 \alpha_3 r_1^n + \alpha_4 r_2^n + n\alpha_5 r_2^n$

Example. char. poly is $(t-1)^d$. From above, $\exists\ \alpha_1, \dots, \alpha_d$ s.t.

$$a_n = \alpha_1 + \alpha_2 n + \dots + \alpha_d n^{d-1} \quad (n \geq 0)$$

i.e., a_n is given by polynomial.

Let T = translation operator on sequences $(Ta)_n = a_{n+1}$

Then $a_n = c_1 a_{n-1} + \dots + c_d a_{n-d}$ (for $n \geq d$) can be written as

$$T^d a = c_1 T^{d-1} a + \dots + c_d a \qquad a = (a_n)_{n \geq 0}$$

(evaluate at position $n-d$: $(T^d a)_{n-d} = c_1 (T^{d-1} a)_{n-d} + \dots + c_d a_{n-d}$

$a_n = c_1 a_{n-1} + \dots + c_d a_{n-d}$)

$$(T^d - c_1 T^{d-1} - \dots - c_d)\, a = 0$$

$$(T - r_1) \cdots (T - r_d)\, a = 0$$

Prop. Given sequence $a = (a_n)_{n \geq 0}$, $\exists$ polynomial $p(n)$ of degree $\leq d-1$ s.t. $a_n = p(n)\ \forall n \geq 0 \iff (T-1)^d a = 0$.

Pf. ($\Leftarrow$) From before: $(T-1)^d a = 0 \Rightarrow a$ satisfies relation of order d whose char. poly is $(t-1)^d$.

($\Rightarrow$) Suppose $p(n) = p_{d-1} n^{d-1} + \dots + p_0$ poly of deg $\leq d-1$.

$$(T-1)(p(n)) = (p(n+1) - p(n))$$

Note: $p(n+1)$ also has $\deg \leq d-1$, coeff of n^{d-1} is p_{d-1}

$\Rightarrow p(n+1) - p(n)$ has $\deg \leq d-2$.

By induction, $(T-1)^d (p(n)) = (0)$

Base case: $d=1$: $p(n)$ = constant $\square$

Rmk. $T-1$ is discrete version of derivative

discrete version of integration S is $(Sa)_n = a_0 + a_1 + \cdots + a_{n-1}$

$\Rightarrow$ $(T-1)Sa = a$, $S(T-1)a = a + \text{const.}$

(Discrete) Fund. Thm of Calculus (finite difference calculus)

Non-homog. recurrence relations

Ex. Consider $a_n = ca_{n-1} + d$, $(n \geq 1)$ c, d constants.

$$a_n = ca_{n-1} + d = c(ca_{n-2} + d) + d = c^2 a_{n-2} + (c+1)d$$
$$= c^2(ca_{n-3} + d) + (c+1)d = c^3 a_{n-3} + (c^2 + c + 1)d$$
$$\cdots = c^n a_0 + (c^{n-1} + c^{n-2} + \cdots + 1)d$$

If $c = 1$, $a_n = a_0 + nd$

If $c \neq 1$, $1 + c + c^2 + \cdots + c^{n-1} = \dfrac{1-c^n}{1-c}$, $a_n = c^n a_0 + \dfrac{1-c^n}{1-c} d$.

Different way:

$$a_n = ca_{n-1} + d$$
$$a_{n-1} = ca_{n-2} + d$$
$$a_n - a_{n-1} = c(a_{n-1} - a_{n-2})$$

$\Rightarrow$ $a_n = (c+1)a_{n-1} - ca_{n-2}$

Char. poly is $t^2 - (c+1)t + c = (t-c)(t-1)$

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a_n satisfies linear recurrence relation w/ polynomial offset.

Then $a_n - a_{n-1}$ satisfies relation w/ polynomial offset of smaller degree.

Ex.

$$a_n = c a_{n-1} + d_1 n + d_2, \quad (n \geq 1) \qquad c,\ d_1,\ d_2 \text{ constants}$$

$$a_{n-1} = c a_{n-2} + d_1 (n-1) + d_2$$

$$a_n - a_{n-1} = c(a_{n-1} - a_{n-2}) + d_1$$

$$\Rightarrow a_n = (c+1) a_{n-1} - c a_{n-2} + d_1 \qquad (n \geq 2)$$

$$a_{n-1} = (c+1) a_{n-2} - c a_{n-3} + d_1$$

$$a_n - a_{n-1} = (c+1)(a_{n-1} - a_{n-2}) - c(a_{n-2} - a_{n-3})$$

$$\Rightarrow \quad a_n = (c+2) a_{n-1} - (2c+1) a_{n-2} + c a_{n-3} \quad (\text{for } n \geq 3)$$

char. poly is $t^3 - (c+2) t^2 + (2c+1) t - c = (t-c)(t-1)^2$

If $c \neq 1$, $a_n = \alpha_1 c^n + \alpha_2 n + \alpha_3$

If $c = 1$, $a_n = \alpha_1 n^2 + \alpha_2 n + \alpha_3$