Formal Power Series

Def. A formal power series is an expression $A(x) = \sum_{n \geq 0} a_n x^n$ (x = variable, a_n = scalars)

For examples, $B(x) = \sum_{n \geq 0} b_n x^n$. $[x^n] A(x) := a_n$

EQUALITY $A(x) = B(x) \iff a_n = b_n \ \forall n \geq 0$

ADDITION $A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n$

- commutative: $A(x) + B(x) = B(x) + A(x)$
- associative: $(A(x) + B(x)) + C(x) = A(x) + (B(x) + C(x))$
- identity: $0 + A(x) = A(x)$ $\left[0 = \sum_{n \geq 0} 0 \cdot x^n \right]$

MULTIPLICATION $A(x)B(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n$

$$\left[\begin{aligned} &A(x) = a_0 + a_1 x + a_2 x^2 + \cdots, \quad B(x) = b_0 + b_1 x + b_2 x^2 + \cdots \\ &A(x)B(x) = a_0 (b_0 + b_1 x + b_2 x^2 + \cdots \quad = a_0 b_0 + (a_0 b_1 + a_1 b_0) x \\ &\qquad + a_1 x (b_0 + b_1 x + b_2 x^2 + \cdots \qquad + (a_0 b_2 + a_1 b_1 + a_2 b_0) x^2 \\ &\qquad + a_2 x^2 (b_0 + b_1 x + b_2 x^2 + \cdots) \qquad + \cdots \end{aligned} \right]$$

- commutative: $A(x)B(x) = B(x)A(x)$
- associativity: $(A(x)B(x))C(x) = A(x)(B(x)C(x))$
- identity: $1 \cdot A(x) = A(x)$ $[1 = 1 + 0x + 0x^2 + \cdots]$
- distributive: $A(x)(B(x) + C(x)) = A(x)B(x) + A(x)C(x)$
- additive inverses: Define $-A(x) = \sum_{n \geq 0} (-a_n) x^n$. Then $A(x) + (-A(x)) = 0$.

$\Longrightarrow$ Formal power series forms a commutative ring

Example $A(x) = B(x) = \sum_{n\geq 0} x^n$.

$$A(x)+B(x) = \sum_{n\geq 0} 2x^n, \quad A(x)B(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} 1\cdot 1\right)x^n = \sum_{n\geq 0}(n+1)x^n.$$

Definition. $A(x)$ is invertible if $\exists\, B(x)$ s.t. $A(x)B(x) = 1$.

If $B(x)$ exists, then it is unique, and $B(x)A(x) = 1$

Notation: $A(x)^{-1} = B(x) = \dfrac{1}{A(x)}$.

Example ① $A(x) = \sum_{n\geq 0} x^n$. $B(x) = 1-x$.

$$A(x)B(x) = (1+x+x^2+\cdots)\cdot 1 - (1+x+x^2+\cdots)\,x$$
$$= (1+\cancel{x}+\cancel{x^2}+\cancel{\cdots}) - (\cancel{x}+\cancel{x^2}+\cancel{x^3}+\cdots) = 1$$

$$A(x)B(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n = 1$$

if $n=0$: $\sum_{i=0}^{0} a_i b_{0-i} = a_0 b_0 = 1$

if $n>0$: $\sum_{i=0}^{n} a_i b_{n-i} = 1-1 = 0$

$\Longrightarrow$ $A(x)^{-1} = B(x)$, so $B(x)^{-1} = A(x)$

$\Rightarrow$ $\dfrac{1}{1-x} = \sum_{n\geq 0} x^n$ [geometric series]

② $A(x) = x$ is not invertible: For any $B(x)$, $xB(x)$ has no constant term, so $xB(x) \neq 1$ for all $B(x)$.

$$\boxed{\textbf{Thm } A(x) \text{ is invertible} \iff a_0 \neq 0.}$$

Pf $A(x) = \sum_{n\geq 0} a_n x^n$. Suppose $a_0 \neq 0$. Want to find inverse.

We want to solve for $b_0, b_1, \ldots$ s.t. $A(x)B(x) = 1$.

by comparing coeff, translates to system of eqns

$$a_0 b_0 = 1 \Rightarrow b_0 = 1/a_0$$
$$a_0 b_1 + a_1 b_0 = 0 \Rightarrow b_1 = \frac{-a_1 b_0}{a_0} = \frac{-a_1}{a_0^2}$$
$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \Rightarrow b_2 = \frac{-a_1 b_1 - a_2 b_0}{a_0}$$
$$a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 = 0 \Rightarrow b_3 = \frac{-a_1 b_2 - a_2 b_1 - a_3 b_0}{a_0}$$
$$\vdots$$

Know: $a_0 \neq 0$. all a_i are given.

structure of eqns tells us we can solve for b_n iteratively:

$$b_n = -\frac{1}{a_0}(a_1 \underline{b_{n-1}} + a_2 \underline{b_{n-2}} + \dots + a_n \underline{b_0})$$

we found in previous step

Define $B(x) := \sum_{n \geq 0} b_n x^n$ where b_n solved in this way.

Now suppose $A(x)$ invertible, $A(x)B(x) = 1 \Rightarrow a_0 b_0 = 1 \Rightarrow a_0 \neq 0$. $\square$

CONVERGENCE

Def. Let $A_0(x), A_1(x), A_2(x), \dots$ be formal power series.

This sequence converges if for all $n \geq 0$, the sequence

$[x^n]A_0(x), [x^n]A_1(x), [x^n]A_2(x), \dots$ is eventually constant.

i.e., for each n, $\exists\, C_n$ s.t. $[x^n]A_i(x) = [x^n]A_{i+1}(x)$ if $i \geq C_n$.

If this converges, let a_n be the constant value of $[x^n]A_i(x)$

Define $A(x) = \sum_{n \geq 0} a_n x^n$. We write $\boxed{\lim_{i \to \infty} A_i(x) = A(x)}$.

Example ① Define $A_i(x) = 1 + x + x^2 + \dots + x^i = \sum_{n=0}^{i} x^n$

Claim: $A_i(x)$ converges, $\lim_{i \to \infty} A_i(x) = \sum_{n \geq 0} x^n$.

$$[x^n]A_i(x) = \begin{cases} 1 & \text{if } n \leq i \\ 0 & \text{if } n > i \end{cases}, \quad [x^n]A_0(x), [x^n]A_1(x), \dots$$
$$= \underbrace{0, 0, \dots, 0}_{n}, \underbrace{1, 1, 1, 1, \dots}_{\text{constant}}$$

② $A_i(x) = \frac{1}{i+1} + 0x + 0x^2 + \cdots$ This does not converge.

$[x^0]\, A_i(x) = \frac{1}{i+1}$ not eventually constant.

Lemma. Assume $\lim_{i\to\infty} A_i(x) = A(x)$, $\lim_{i\to\infty} B_i(x) = B(x)$.

Then: $\lim_{i\to\infty} (A_i(x) + B_i(x)) = A(x) + B(x)$, $\lim_{i\to\infty} \left(A_i(x) B_i(x)\right) = A(x) B(x)$

Back to invertibility: at each step, we have solved for $b_0, \ldots, b_n$ s.t. if we set $B_i(x) = \sum_{n=0}^{i} b_n x^n$, then $A(x) B_i(x) = 1 + 0x + \cdots + 0x^i + \underbrace{?}_{\text{terms of degree } \geq i+1}$

$1 = \lim_{i\to\infty} A(x) B_i(x) = A(x) \lim_{i\to\infty} B_i(x) = A(x)\, B(x)$.

INFINITE SUMS + PRODUCTS

Given $A_0(x), A_1(x), \ldots$ formal power series.

Define $\sum_{i\geq 0} A_i(x) := \lim_{i\to\infty} \sum_{j=0}^{i} A_j(x)$ if it exists

$\prod_{i\geq 0} A_i(x) := \lim_{i\to\infty} \prod_{j=0}^{i} A_j(x)$ if it exists.

Lemma from before gives

Lemma. Assuming all limits exist, we have:

$$\sum_{i\geq 0} A_i(x) + \sum_{i\geq 0} B_i(x) = \sum_{i\geq 0} \left(A_i(x) + B_i(x)\right)$$

$$\left(\prod_{i\geq 0} A_i(x)\right)\left(\prod_{i\geq 0} B_i(x)\right) = \prod_{i\geq 0} \left(A_i(x) B_i(x)\right).$$

Example. $A_i(x) = x^i$, $\sum_{j=0}^{i} A_j(x) = 1 + x + x^2 + \cdots + x^i$

$\sum_{i\geq 0} A_i(x) = \lim_{i\to\infty} (1 + x + \cdots + x^i) = \sum_{n\geq 0} x^n$

COMPOSITION

$A(x)$, $B(x)$ formal power series, $a_0 = 0$.

The <u>composition</u> is $(B \circ A)(x) = B(A(x)) = \sum_{n \geq 0} b_n A(x)^n$

Make sense? i.e., does $\lim_{i \to \infty} \sum_{j=0}^{i} b_j A(x)^j$ exist?

Yes: since $a_0 = 0$, $[x^n] A(x)^j = 0$ if $n < j$.

$\Longrightarrow$ for $i \geq n$, $[x^n] \sum_{j=0}^{i} b_j A(x)^j = [x^n] \sum_{j=0}^{n} b_j A(x)^j$

<u>Example</u> ① $d > 0$, $A(x) = x^d$, $B(x) = \sum_{n \geq 0} x^n$.

$$B(A(x)) = \sum_{n \geq 0} (x^d)^n = \sum_{n \geq 0} x^{dn}$$

Consider identity $(1-x)B(x) = 1$. compose w/ $A(x)$:

$$(1-x^d)B(x^d) = 1 \Longrightarrow \frac{1}{1-x^d} = \sum_{n \geq 0} x^{dn}.$$

More generally, if $a_0 = 0$, $\frac{1}{1-A(x)} = \sum_{n \geq 0} A(x)^n$.

② $A(x) = 1$, $B(x) = \sum_{n \geq 0} x^n$, $B(A(x))$ is <u>undefined</u>.

~~if it were defined, then we would have~~ $\cancel{\frac{1}{1-A(x)} \cdot B(A(x)) = 1}$

$\cancel{1/0 \to}$

DERIVATIVES

$A(x)$ formal power series

The <u>derivative</u> is $(DA)(x) = \sum_{n \geq 0} n a_n x^{n-1} = \sum_{n \geq 0} (n+1) a_{n+1} x^n$

- $D(A+B) = DA + DB$
- $D(AB) = (DA)B + A(DB)$

$a_0 = 0$ $\cdot\ D(B \circ A) = (DA)(DB \circ A)$

$a_0 \neq 0$ $\cdot\ D(1/A) = -DA/A^2$

$\cdot\ D(A^n) = n(DA)A^{n-1}$

Example. (1) Consider $\frac{1}{1-x} = \sum_{n \geq 0} x^n$. Apply D:

$$\frac{1}{(1-x)^2} = \sum_{n \geq 0} n x^{n-1} = \sum_{n \geq 0} (n+1) x^n$$

(2) How to simplify $B(x) = \sum_{n \geq 0} n x^n$?

(a) $B(x) = \sum_{n \geq 0} (n+1)x^n - \sum_{n \geq 0} x^n = \frac{1}{(1-x)^2} - \frac{1}{1-x} = \frac{1-(1-x)}{(1-x)^2} = \frac{x}{(1-x)^2}$.

(b) $B(x) = x \sum_{n \geq 0} n x^{n-1} = x D\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2}$.

$$[x^n] A(x) = \frac{(D^n A)(0)}{n!}$$ $[n! = n(n-1)(n-2) \cdots 1]$