12-fold way

$f: \{k \text{ balls}\} \to \{n \text{ boxes}\}$

Conditions on f:
1. arbitrary
2. f is injective (one-to-one)
3. f is surjective (onto)

Conditions on balls
1. balls are distinguishable
2. balls are indistinguishable

Conditions on boxes
1. boxes are distinguishable
2. boxes are indistinguishable.

There is an action of symmetric group $\mathfrak{S}_k$ on k letters on the set of functions: relabels balls, so indistinguishable balls we count $\mathfrak{S}_k$-orbits of functions rather than functions

Similarly, have action of $\mathfrak{S}_n$ on functions by relabeling boxes

balls/boxes	f arbitrary	f injective	f surjective
dist / dist			
indist / dist			
dist / indist		*	
indist / indist		*	

* boring: answer is 1 if $n \geq k$
0 if $n < k$

f bijective? special case of f injective when $n = k$

Compositions n, k positive integers

Defn. A sequence of non-negative integers $(a_1, \ldots, a_n)$ is a *weak composition* of k if $a_1 + \cdots + a_n = k$.

It is a *composition* if all a_i are positive.

Problem this addresses: boxes are distinguishable
balls are indistinguishable

$a_i = \#$ balls in box ith box.

weak composition: no condition on f

Composition: f required to be surjective.

Thm. The # of weak compositions of k into n parts is

$$\binom{n+k-1}{k} = \binom{n+k-1}{n-1}.$$

Pf. Consider the expansion

$$\frac{1}{(1-x)^n} = \left(\sum_{a\geq 0} x^a\right)^n = \left(\sum_{a_1\geq 0} x^{a_1}\right)\left(\sum_{a_2\geq 0} x^{a_2}\right)\cdots\left(\sum_{a_n\geq 0} x^{a_n}\right)$$

$$= \sum_{(a_1,\ldots,a_n)\in\mathbb{Z}^n_{\geq 0}} x^{a_1+\cdots+a_n}$$

$$[x^k]\frac{1}{(1-x)^n} = \#\text{ weak compositions of } k \text{ into } n \text{ parts}$$

$$= \binom{n+k-1}{k}. \quad \text{(binomial thm)} \qquad \square$$

Rmk. $\binom{n+k-1}{k} = \#$ of multisets of $[n]$ of size k

$\rightsquigarrow$ suggests bijection between $\{$weak compositions of k into n parts$\}$ and $\{$multisets of $[n]$ of size $k\}$

Given a weak composition $(a_1, \ldots, a_n)$ of k

$\rightsquigarrow$ multiset of $[n]$ where i gets chosen a_i times

Ex. Distribute 20 pieces of candy to 4 children.

Ways to do this are weak compositions of 20 into 4 parts

$$\# = \binom{20+4-1}{20} = \binom{23}{20}$$

Compositions = ways to distribute s.t. every child receives at least 1 candy. First give each child 1 candy

Then distribute the rest arbitrarily.

$\#$ ways = weak compositions of 16 into 4 parts

$$= \binom{16+4-1}{16} = \binom{19}{16}.$$

Cor. $\#$ compositions of k into n parts $= \binom{k-1}{n-1}$.

Pf. Bijection $\left\{ \begin{array}{c} \text{weak compositions of} \\ k-n \text{ into } n \text{ parts} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{compositions of } k \\ \text{into } n \text{ parts} \end{array} \right\}$

$$(a_1 - 1, \ldots, a_n - 1) \longleftarrow (a_1, \ldots, a_n)$$

$$(b_1, \ldots, b_n) \longmapsto (b_1 + 1, \ldots, b_n + 1)$$

$\Rightarrow$ $\#$ compositions of k into n parts $= \binom{(k-n)+n-1}{k-n} = \binom{k-1}{k-n} = \binom{k-1}{n-1}$ $\square$

Generating functions:

$$\frac{x^n}{(1-x)^n} = \left(\sum_{a \geq 1} x^a\right)^n = \left(\sum_{a_1 \geq 1} x^{a_1}\right) \cdots \left(\sum_{a_n \geq 1} x^{a_n}\right) = \sum_{(a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 1}} x^{a_1 + \cdots + a_n}$$

$\Rightarrow$ $\#$ compositions of k into n parts $= [x^k] \dfrac{x^n}{(1-x)^n} = [x^{k-n}] \dfrac{1}{(1-x)^n} = \binom{k-1}{n-1}.$

<u>Cor</u>. # compositions of $k = 2^{k-1}$

<u>Pf</u>. # compositions of $k = \sum_{n=1}^{k}$ # compositions of k into n parts $= \sum_{n=1}^{k} \binom{k-1}{n-1}$

$m = n-1$

$$= \sum_{m=0}^{k-1} \binom{k-1}{m} = 2^{k-1}.$$

$\square$

<u>Rmk</u>. # compositions of k = # subsets of $[k-1]$

Is there a direct bijection?

$$\left\{\begin{array}{c}\text{compositions} \\ \text{of } k\end{array}\right\} \longrightarrow \left\{\text{subsets of } [k-1]\right\}$$

$$(a_1, \dots, a_r) \longmapsto \{a_1, a_1+a_2, a_1+a_2+a_3, \dots, a_1+\dots+a_{r-1}\}$$

$$(s_1, s_2-s_1, s_3-s_2, \dots, s_p-s_{p-1}, k-s_p) \longleftarrow \{s_1, \dots, s_p\}$$ assume $s_1 < s_2 < \dots < s_p$