Math 264C, Spring 2021
Homework 1

(1) Let W be the group of $n \times n$ signed permutation matrices, i.e., $n \times n$ matrices with exactly one nonzero entry in each row and column, and that entry can be 1 or -1. (So $|W| = 2^n n!$.)
   (a) The adjacent transpositions can be completed to a set of Coxeter generators for W by adding 1 element. Find such an element and describe the length function with respect to these generators.
   (b) What is the longest element in W?
   (c) Which elements are reflections?
   (d) Using the adjacent transpositions in (a), the symmetric group $\mathfrak{S}_n$ is a parabolic subgroup of W. Describe the minimal length coset representatives.

(2) Let V be the geometric representation of (W, S) and let $R = \mathrm{Sym}(V) = \mathbf{R}[\alpha_s \mid s \in S]$ be the ring of polynomials in the α_s.
   (a) For $s \in S$ and $f \in R$, show that $f - sf$ is divisible by α_s.
   In particular, we can define $\partial_s \colon R \to R$ by
$$\partial_s(f) = \frac{f - sf}{\alpha_s}.$$
   (b) Show that $\partial_s^2 = 0$.
   (c) For $s, t \in S$ where $m(s,t) \leq 3$, show that $\partial_s \partial_t \partial_s \cdots = \partial_t \partial_s \partial_t \cdots$ where each side is a product of size $m(s,t)$.
   [This holds in general as long as $m(s,t) < \infty$, but requires a lot more effort.]

(3) Given a permutation $\sigma \in \mathfrak{S}_n$, define $r_\sigma(p,q) = |\{i \leq p \mid \sigma(i) \leq q\}|$. Show that $\sigma \leq \tau$ (Bruhat order) if and only if $r_\sigma(p,q) \geq r_\tau(p,q)$ for all p, q.

(4) Let $W = \mathfrak{S}_n$ and $W_P = \mathfrak{S}_k \times \mathfrak{S}_{n-k}$.
   (a) For a minimal length coset representative $\sigma \in W^P$, we have $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$. For $\sigma, \tau \in W^P$, show that $\sigma \leq \tau$ (Bruhat order) if and only if $\sigma(i) < \tau(i)$ for $i = 1, \ldots, k$.
   (b) Consider integer sequences λ (partitions) such that $n - k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0$. Define $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i. Show that W^P is order-isomorphic to partitions (with our restrictions).