Math 264C, Spring 2021
Homework 2

(1) Let the alternating group $\mathfrak{A}_n$ be the subgroup of $\mathfrak{S}_n$ consisting of even permutations. Find a minimal set of generators for $\mathbf{C}[x_1, \dots, x_n]^{\mathfrak{A}_n}$ and show that they are not algebraically independent.

(2) (a) Let $\mathbf{F}_q$ be a finite field with q elements and let $G = \mathbf{GL}_n(\mathbf{F}_q)$ be the group of $n \times n$ invertible matrices with entries in $\mathbf{F}_q$. Let $V = \mathbf{F}_q^n$ be the space of column vectors, so that we have an action of G on V (this is a representation over $\mathbf{F}_q$) and hence an action on $\mathrm{Sym}(V) \cong \mathbf{F}_q[x_1, \dots, x_n]$.
Show that the ring of invariants $\mathrm{Sym}(V)^G$ is generated by algebraically independent generators of degrees $q^n - q^{n-1}, q^n - q^{n-2}, \dots, q^n - 1$.
(I have a writeup here: https://concretenonsense.wordpress.com/2011/03/17/dickson-invariants/. Try to write it up from your own understanding.)

(b) Call a finite-order linear operator A a **pseudoreflection** if $\mathrm{rank}(A - I) = 1$. Show that over a field of positive characteristic, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is a pseudoreflection. Furthermore, show that every pseudoreflection is either diagonalizable or equal to a block sum of this matrix with the identity with respect to some choice of basis.

(c) Let G be a finite subgroup of $\mathbf{GL}(V)$. A theorem of Serre says that if $\mathrm{Sym}(V)^G$ is generated by algebraically independent polynomials, then G is generated by pseudoreflections (we have proven this when the field has characteristic 0)[1]. Verify this theorem for the example in (a) by showing that $\mathbf{GL}_n(\mathbf{F}_q)$ is generated by pseudoreflections.

(3) Consider Theorem 3.24 for $W = \mathfrak{S}_n$ acting on $\mathbf{C}^n$ (rather than $\mathbf{C}^{n-1}$). The multiplicity of the eigenvalue 1 of a permutation is the number of cycles, so we get an identity

$$t(t+1)\cdots(t+n-1) = \sum_{\sigma \in \mathfrak{S}_n} t^{\#\mathrm{cycles}(\sigma)}.$$

What is the corresponding analogue for the hyperoctahedral group $W(\mathrm{B}_n)$?

(4) Open ended: investigate one of the items about the Valentiner group https://en.wikipedia.org/wiki/Valentiner_group and write a short summary explaining it and any relevant terminology (approx. 1 paragraph is fine). Or you can pick a different complex reflection group between numbers 23 and 37 in https://en.wikipedia.org/wiki/Complex_reflection_group and discuss some aspect not addressed in the lectures.

[1] The converse fails: there exist finite groups generated by pseudoreflections in positive characteristic whose ring of invariants is not generated by algebraically independent polynomials.