Coxeter Groups

<u>Def</u>. S = finite set, $m: S\times S \to \mathbb{Z}_{>0}\cup\{\infty\}$ s.t.

- $m(s,s)=1 \quad \forall s\in S$
- $m(s,t)=m(t,s) \quad \forall s,t$
- $m(s,t)\geq 2$ if $s\neq t$.

Coxeter System

The <u>Coxeter group</u> W is group generated by S s.t.

$$(st)^{m(s,t)}=1 \qquad \forall s,t\in S \text{ s.t. } m(s,t)<\infty.$$

$\Rightarrow s^2=1$ for all $s\in S$, W generated by involutions

(W,S) will denote all of this data, $|S|$ is <u>rank</u> of (W,S).

We have sign homomorphism $W\to\{1,-1\}$

$s\mapsto -1$ for all $s\in S$

<u>Ex</u>. $S=\{s,t\}$, $m(s,t)<\infty$. W has presentation:

$$W\cong\langle s,t \mid s^2=t^2=(st)^{m(s,t)}=1\rangle$$

is dihedral group of order $2m(s,t)$, symmetries of regular m-gon.

If $m=\infty$, W is "infinite dihedral group"

<u>Ex</u>. $S=\{s_1,\dots,s_{n-1}\}$ $\quad m(s_i,s_j)=\begin{cases}3 & \text{if } |i-j|=1\\ 2 & \text{if } |i-j|>1\end{cases}$ Coxeter system A_{n-1}

Let $\mathfrak{S}_n$ = symmetric group on n letters. (i,j) = transposition swapping i & j.

Define $\varphi: W\to\mathfrak{S}_n$, $\varphi(s_i)=(i,i+1)$

- $\varphi(s_i)^2=1$
- $\varphi(s_i)\varphi(s_{i+1})=(i,i+1)(i+1,i+2)=(i\ \ i+1\ \ i+2)$ has order 3
- if $|i-j|>1$, $(i,i+1)$, $(j,j+1)$ commute, $(\varphi(s_i)\varphi(s_j))^2=1$.

<u>Ex</u>. $W\subseteq GL_n(\mathbb{R})$ generated by reflections, W finite

$\Rightarrow W\cong$ Coxeter group.

Geometric representation. $V = \mathbb{R}$-vector space w/ basis $\{\alpha_s \mid s \in S\}$

B = symmetric bilinear form on V given by

$$B(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m(s,t)}\right) & \text{if } m(s,t) < \infty \\ -1 & \text{if } m(s,t) < \infty \end{cases}.$$

α_s are unit vectors, for each $s \in S$, define $\sigma_s \in GL(V)$ by

$$\sigma_s(v) = v - 2B(\alpha_s, v)\alpha_s.$$

Note: $\sigma_s(\alpha_s) = -\alpha_s$.

Lemma. For $s \in S$, $v, w \in V$, $B(v,w) = B(\sigma_s v, \sigma_s w)$

Pf. $B(\sigma_s v, \sigma_s w) = B(v - 2B(\alpha_s, v)\alpha_s, w - 2B(\alpha_s, w)\alpha_s)$

$= B(v,w) - 2B(v, \alpha_s)B(\alpha_s, w) - 2B(\alpha_s, v)B(\alpha_s, w) + 4B(\alpha_s, v)B(\alpha_s, w)\cancel{B(\alpha_s, \alpha_s)}$

$= B(v,w)$. $\square$

Lemma. For $s,t \in S$, $\sigma_s\sigma_t$ has order $m(s,t)$.

Pf. • If $s = t$, then for any $v \in V$,

$\sigma_s^2(v) = \sigma_s(v - 2B(\alpha_s, v)\alpha_s)$

$= v - 2\cancel{B(\alpha_s, v)}\alpha_s + 2\cancel{B(\alpha_s, v)}\alpha_s = v \Rightarrow \sigma_s^2 = 1$.

• $s \neq t$. $m = m(s,t)$. $U = \operatorname{Span}\{\alpha_s, \alpha_t\}$.

• Suppose $m < \infty$. Claim: $B|_U$ is positive definite.

Pf of claim. Let $v = c_s\alpha_s + c_t\alpha_t$.

$B(v,v) = c_s^2 - 2c_sc_t\cos\left(\frac{\pi}{m}\right) + c_t^2$

$= c_s^2 - 2c_sc_t\cos\left(\frac{\pi}{m}\right) + c_t^2\left(\cos^2\left(\frac{\pi}{m}\right) + \sin^2\left(\frac{\pi}{m}\right)\right)$

$= \left(c_s - c_t\cos\left(\frac{\pi}{m}\right)\right)^2 + \left(c_t\sin\left(\frac{\pi}{m}\right)\right)^2.$

$m \geq 2 \Rightarrow \sin\left(\frac{\pi}{m}\right) > 0$. If $c_t \neq 0$, then $B(v,v) > 0$.

If $c_t = 0$, $B(v,v) = c_s^2 > 0$ (if $v \neq 0$). B pos. def. on U.

Define $U^\perp = \{ v \in V \mid B(v, \alpha_s) = B(v, \alpha_t) = 0 \}$.

$$U^\perp \cap U = 0 \Rightarrow V = U^\perp \oplus U$$

σ_s, σ_t are identity on $U^\perp$, so $\text{order}(\sigma_s \sigma_t) = \text{order}(\sigma_s \sigma_t|_U)$.

$$B(\alpha_s, \alpha_t) = -\cos\left(\frac{\pi}{m}\right) = \cos\left(\pi - \frac{\pi}{m}\right). \quad \theta := \pi - \frac{\pi}{m}.$$

Have isometry $U \longrightarrow \mathbb{R}^2$ s.t. $\alpha_s \to (1,0)$, $\alpha_t \to (\cos\theta, \sin\theta)$

$$\sigma_s \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_t \to \begin{pmatrix} 1 - 2\cos^2\theta & -2\cos\theta\sin\theta \\ -2\cos\theta\sin\theta & 1 - 2\sin^2\theta \end{pmatrix} = \begin{pmatrix} -\cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & \cos(2\theta) \end{pmatrix}$$

double angle formulas

$$\sigma_s\sigma_t \to \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} = \text{rotation by } -2\theta = \frac{2\pi}{m}.$$

$\Rightarrow \sigma_s \sigma_t$ has order m.

- $m = \infty$, $B(\alpha_s, \alpha_t) = -1$.

$$\sigma_s \sigma_t(\alpha_s) = \sigma_s(\alpha_s + 2\alpha_t) = -\alpha_s + 2\alpha_t + 4\alpha_s = 3\alpha_s + 2\alpha_t.$$
$$= 2u + \alpha_s$$

$u = \alpha_s + \alpha_t$. Note: $\sigma_s(u) = \sigma_t(u) = u$

$(\sigma_s \sigma_t)^k(\alpha_s) = 2ku + \alpha_s \Rightarrow \sigma_s \sigma_t$ has infinite order. $\square$

Cor. We have homomorphism $\sigma : W \longrightarrow GL(V)$ given by $\sigma(s) = \sigma_s$, $s \in S$.

Ex. (A_{n-1}) $\alpha_i = \alpha_{s_i}$. $B(\alpha_i, \alpha_j) = \begin{cases} -1/2 & \text{if } |i-j| = 1 \\ 1 & \text{if } i = j \\ 0 & \text{else} \end{cases}$

Consider hyperplane H in $\mathbb{R}^{n-1}$ given by $\{(x_1, \dots x_n) \mid x_1 + \dots + x_n = 0\}$

$V \cong H$, $\alpha_i \to \frac{1}{\sqrt{2}}(e_i - e_{i+1})$, σ_i becomes the matrix on $(x_1, \dots, x_n)$ $\square$

swaps x_i and x_{i+1}.