Polynomial Invariants

$\mathbb{k}$ = field, V = vector space of dimension n, pick basis $x_1, \ldots x_n$ for V.

$$A = \mathbb{k}[x_1, \ldots, x_n] = \mathrm{Sym}(V) = \bigoplus_{d \geq 0} \mathrm{Sym}^d(V)$$

G = finite group acting on V, G also acts on each $\mathrm{Sym}^d V$, and A.

Since $(V^*)^* = V$, A = ring of polynomial functions on V^*

An <u>ideal</u> $I \subset A$ (nonempty) s.t. if $f, g \in I \Rightarrow f + g \in I$ and if $f \in I$, $g \in A$, then $fg \in I$. A set S generates I if

$$I = \{ \sum_{s \in S} c_s\, s \text{ (finite sum)} \mid c_s \in A \}$$

A finitely generated $\mathbb{k}$-algebra is a quotient ring of A by an ideal

<u>Thm</u> (Hilbert basis thm). Let R be a finitely generated $\mathbb{k}$-algebra. Every ideal of R has a finite generating set.

More generally, if M is a finitely generated R-module, then every submodule of M is finitely generated.

<u>Gradings</u> Pick positive integers $d_1, \ldots, d_n$, set $\deg(x_i) = d_i$

$\Rightarrow$ decomposition $A = \bigoplus_{d \geq 0} A_d$ where A_d = $\mathbb{k}$-span of all $x_1^{p_1} \cdots x_n^{p_n}$ s.t. $d_1 p_1 + \cdots + d_n p_n = d$. (<u>Note</u>: $A_0 = \mathbb{k}$ spanned by constant poly)

A module M is <u>graded</u> if $M = \bigoplus_{d \geq 0} M_d$ s.t. $\forall f \in A_d$, $m \in M_e$, $fm \in M_{d+e}$.

An element of M_d is called <u>homogeneous</u>.

An ideal I is <u>homogeneous</u> if $I = \bigoplus_{d \geq 0} (I \cap A_d)$

In this case, I is graded w/ $I_d = I \cap A_d$.

Homogeneous modules always have generating sets consisting of homogeneous elements

If M is graded module, its Hilbert series is

$$H_M(t) = \sum_{d \geq 0} (\dim_{\Bbbk} M_d)\, t^d .$$

Given graded modules M, N, their tensor product $^{(/\Bbbk)}$ is graded w/

$$(M \otimes_{\Bbbk} N)_d = \bigoplus_{e=0}^{d} M_e \otimes N_{d-e} .$$

$$H_{M \otimes N}(t) = H_M(t)\, H_N(t).$$

Important case: $\Bbbk[x_1, \ldots, x_n] = \Bbbk[x_1] \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} \Bbbk[x_n]$

$$H_{\Bbbk[x_1, \ldots, x_n]}(t) = \prod_{i=1}^{n} \frac{1}{1 - t^{\deg(x_i)}} .$$

$f_1 \ldots f_k \in A$ are algebraically independent if, for any nonzero polynomial $h(y_1, \ldots y_k)$ (y's are new variables), we have $h(f_1, \ldots f_k) \neq 0$.

Any alg. ind. set becomes alg. ind. set in $\mathrm{Frac}(\Bbbk[x_1, \ldots x_n]) = \Bbbk(x_1, \ldots, x_n)$ which can be extended to a transcendence basis over $\Bbbk$

$\Rightarrow$ alg. ind. sets have size $\leq n$

Molien's formula Let $\Bbbk = \mathbb{C}$.

Lemma. V = finite dim vector space, w/ finite group G acting on it.

Define $\varphi : V \to V$ by $\varphi(g) = \frac{1}{|G|} \sum_{g \in G} g \cdot v$.

Then
1. φ is a projection, i.e., $\varphi^2 = \varphi$
2. The image of φ is V^G
3. $\dim V^G = \mathrm{Tr}(\varphi) = \frac{1}{|G|} \sum_{g \in G} \mathrm{trace}(g \mid V)$

Pf. First, show $\operatorname{im}\varphi \subseteq V^G$, i.e., $g\varphi(v) = \varphi(v)\ \forall g \in G, v \in V$.

Pick $h \in G$, $v \in V$. Then

$$h \cdot \varphi(v) = \frac{1}{|G|}\sum_{g \in G} \underbrace{h \cdot (g \cdot v)}_{= (h \cdot g) \cdot v} = \frac{1}{|G|}\sum_{g \in G} g \cdot v = \varphi(v).$$

If $w \in V^G$, $\varphi(w) = \frac{1}{|G|}\sum_{g \in G} w = w$, so $\operatorname{im}\varphi = V^G$. $\Rightarrow$ 2

$\Rightarrow \varphi^2 = \varphi$ b/c $\forall v \in V$, $\varphi^2(v) = \varphi(\underbrace{\varphi(v)}_{\in V^G}) = \varphi(v)$. $\Rightarrow$ 1

3 : For any projection, its eigenvalues are either 0 or 1

its rank is multiplicity of 1, which is its trace.

$\Rightarrow$ trace $\varphi = \dim V^G$ $\square$

Thm (Molien's formula). Let G act on V, $A = \operatorname{Sym}(V)$

Let $\rho_V(g)$ be linear operator of g acting on V. Then

$$\sum_{d \geq 0} \dim_{\mathbb{C}} (\operatorname{Sym}^d V)^G t^d = \frac{1}{|G|}\sum_{g \in G} \frac{1}{\det(1 - \rho_V(g)t)}$$

Pf. Pick $g \in G$. Let $z_1, \dots z_n$ be eigenvalues (w/ multiplicity) of $\rho_V(g)$.

Let $v_1, \dots v_n$ be eigenbasis for $\rho_V(g)$. An eigenbasis for g acting on $\operatorname{Sym}^d V$ is $\{v_{i_1} \cdots v_{i_d} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n\}$.

eigenvalue is $z_{i_1} \cdots z_{i_d}$

$$\sum_{d \geq 0} \operatorname{Tr}(g \mid \operatorname{Sym}^d V) t^d = \prod_{i=1}^{n} \frac{1}{1 - z_i t} = \frac{1}{\det(1 - \rho_V(g)t}$$

$$\Rightarrow \sum_{d \geq 0} \dim_{\mathbb{C}} (\operatorname{Sym}^d V)^G t^d = \sum_{d \geq 0} \frac{1}{|G|} \sum_{g \in G} \operatorname{Tr}(g \mid \operatorname{Sym}^d V) t^d = \frac{1}{|G|}\sum_{g \in G} \frac{1}{\det(1 - \rho_V(g)t)} \quad \square$$