Ring of invariants

G = finite group, $G \circlearrowright A = \mathrm{Sym}(V)$, V complex vector space
$\cong \mathbb{C}[x_1, \ldots, x_n]$

$\varphi = \frac{1}{|G|} \sum_{g \in G} g$, Define $f^\# = \varphi(f)$ for $f \in A$ Reynolds operator

If $f_1 \in A^G$, $f_2 \in A$ then $(f_1 f_2)^\# = f_1 f_2^\#$

$$\left(\varphi(f_1 f_2) = \frac{1}{|G|} \sum_{g \in G} g(f_1) g(f_2) = \frac{f_1}{|G|} \sum_{g \in G} g(f_2) = f_1 f_2^\# \right)$$

$\Longrightarrow$ $\# : A \longrightarrow A^G$ is a A^G-module homomorphism.
+ surjective + preserves degrees

Prop. Let $I \subset A$ be ideal generated by A^G.
$$I = \{ \sum_i \alpha_i f_i \mid f_i \in A^G, \alpha_i \in A \}$$
Suppose that $f_1, \ldots, f_k \in A^G$ are positive degree homogeneous elements that generate I. Then $f_1, \ldots, f_k$ generate A^G as a $\mathbb{C}$-algebra.

Pf. Pick $f \in A^G$. We show f is generated by $f_1, \ldots, f_k$ by induction on $\deg(f)$. $f \in A^G \Rightarrow f \in I \Longrightarrow \exists h_1, \ldots, h_k \in A$ homogeneous s.t. $f = h_1 f_1 + \cdots + h_k f_k$. Apply $\#$:

$$f = f^\# = h_1^\# f_1 + \cdots + h_k^\# f_k.$$

Note $\deg h_i^\# + \deg f_i = \deg f$. By induction, $h_i^\#$ is generated by $f_1, \ldots, f_k$ as a $\mathbb{C}$-algebra. Substitute these expressions in for $h_i^\#$ to get that f is generated by $f_1, \ldots f_k$ as $\mathbb{C}$-algebra. $\square$

Cor. A^G is a finitely generated $\mathbb{C}$-algebra.

Pf. I is f.g. by Hilbert basis thm. $\square$

Rmk. Proof extends to any field of characteristic 0.

In general, A^G is f.g. (Noether) let k = any field.

Let t be a new variable. For each i, consider

$$p_i(t) = \prod_{g \in G}(t - g x_i) \in A[t]. \text{ In fact, } p_i(t) \in A^G[t]$$

Let B = k-subalgebra of A^G generated by the coeff. of $p_i(t)$ for all i. By definition, B is finitely generated over k.

Furthermore, A is a finitely generated B-module:

$\{ x_i^j \mid i = 1, \dots, n;\ 0 \le j < |G| \}$ generates A as a B-module:

$x_i^{|G|}$ = linear combination of lower powers of x_i w/ coeffs in B since $p_i(x_i) = 0$.

Next, A^G is a B-submodule of A. Hilbert basis thm $\Rightarrow$ A^G is a f.g. B-module. A set of generators as B-module together w/ generators for B give set of algebra generators for A^G. $\square$

Def. R integral domain, $\mathrm{Frac}(R)$ = field of fractions

Prop $\mathrm{Frac}(A^G) = \mathrm{Frac}(A)^G$.

In particular, if G acts faithfully on V, then $\mathrm{Frac}(A)$ is a degree $|G|$ extension of $\mathrm{Frac}(A^G)$, and tr. deg. $\mathrm{Frac}(A^G) = n$.

Pf. $\mathrm{Frac}(A^G) \subseteq \mathrm{Frac}(A)^G$. Pick $\frac{p}{q} \in \mathrm{Frac}(A)^G$.

Define $p' = \prod_{\substack{g \in G \\ g \neq 1}} gp$. Then pp' is G-invariant. $\frac{p}{q} = \frac{pp'}{qp'}$

& qp' is also Ginvariant $\Rightarrow \frac{p}{q} \in \mathrm{Frac}(A^G)$.

In general, if $G \subset \mathrm{Aut}(K)$, then K is a degree $|G|$ extension of K^G.

Since trdeg is constant within finite extensions,

$\mathrm{trdeg}(\mathrm{Frac}(A^G)) = \mathrm{trdeg}(\mathrm{Frac}\, A) = n$. $\square$