Jacobian

Setup. $h_1, \ldots, h_n \in \mathbb{C}[x_1, \ldots, x_n]$

Jacobian: $J = J(h_1, \ldots h_n) = \det\left(\frac{\partial h_i}{\partial h_j}\right)_{i,j=1,\ldots,n}$

Lemma. $h_1, \ldots h_n$ are algebraically independent $\Leftrightarrow$ $J \neq 0$.

Pf. Suppose $h_1, \ldots h_n$ are dependent. $\exists$ nonzero polynomial $F(y_1, \ldots y_n)$ s.t. $F(h_1, \ldots h_n) = 0$. Pick F w/ smallest possible degree. Apply $\frac{\partial}{\partial x_j}$:

$$0 = \frac{\partial}{\partial x_j} F(h_1, \ldots, h_n) = \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}(h_1, \ldots h_n) \frac{\partial h_i}{\partial x_j}$$

In matrix form:

$$\left[\frac{\partial F}{\partial y_1}(h_1, \ldots, h_n) \quad \cdots \quad \frac{\partial F}{\partial y_n}(h_1, \ldots h_n) \right] \left[\frac{\partial h_i}{\partial x_j} \right]_{i,j=1,\ldots,n} = 0$$

F is not constant $\Rightarrow$ some $\frac{\partial F}{\partial y_j} \neq 0$. By minimality of $\deg F$,

$\frac{\partial F}{\partial y_j}(h_1, \ldots h_n) \neq 0$. $\Rightarrow J = \det\left(\frac{\partial h_i}{\partial x_j}\right) = 0$.

Now suppose $h_1, \ldots h_n$ are algebraically independent. For each i, $\{x_i, h_1, \ldots h_n\}$ is algebraically dependent. For each i, let F_i be a polynomial in $y_0, y_1, \ldots y_n$ of minimal possible degree s.t. $F_i(x_i, h_1, \ldots h_n) = 0$. Apply $\frac{\partial}{\partial x_j}$:

$$0 = \frac{\partial}{\partial x_j} F_i(x_i, h_1, \ldots h_n) = \frac{\partial F_i}{\partial y_0}(x_i, h_1, \ldots h_n)\delta_{ij} + \sum_{k=1}^{n} \frac{\partial F_i}{\partial y_k}(x_i, h_1, \ldots h_n) \frac{\partial h_k}{\partial x_j}$$

In matrix form:

$$\left(\frac{\partial F_i}{\partial y_j}(x_i, h_1, \dots h_n)\right)_{i,j=1,\dots n} \left(\frac{\partial h_i}{\partial x_j}\right)_{i,j=1,\dots n} = -\left(\frac{\partial F_i}{\partial y_0}(x_i, h_1, \dots h_n)\,\delta_{ij}\right)_{i,j=1,\dots n}$$

Since $h_1, \dots h_n$ are alg. ind., F_i is positive degree wrt y_0

$\Rightarrow \frac{\partial F_i}{\partial y_0} \neq 0$ for all i. $\Rightarrow \frac{\partial F_i}{\partial y_0}(x_i, h_1, \dots h_n) \neq 0$ by minimality of $\deg F_i$

diagonal matrix w/ nonzero entries $\Rightarrow \det \neq 0$

$\Rightarrow J \neq 0$

$\square$

As before, $W \subset GL_n\mathbb{C}$ complex reflection group,

$f_1, \dots, f_n \in \mathbb{C}[x_1, \dots, x_n]$ generators for $\mathbb{C}[x_1, \dots, x_n]^W$.

Chevalley $\rightarrow$ $f_1, \dots f_n$ alg. ind. $\xRightarrow{\text{Lemma}}$ $\mathcal{J} = J(f_1, \dots f_n) \neq 0$.

$$\deg \mathcal{J} = \sum_{i=1}^{n} (\deg f_i - 1)$$

Ex. $W = \mathfrak{S}_n \subset GL_n\mathbb{C}$. $f_i = \frac{1}{i}(x_1^i + \dots + x_n^i)$

$$\mathcal{J} = \det\left(\frac{\partial f_i}{\partial x_j}\right) = \det\left(x_j^{i-1}\right) = \begin{pmatrix} 1 & 1 & & 1 \\ x_1 & x_2 & & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & & x_n^{n-1} \end{pmatrix}$$

$$= \prod_{1 \leq i < j \leq n} (x_j - x_i) \neq 0.$$