Solomon's Theorem

$W \subset GL_n\mathbb{C}$ reflection group
$A = \mathbb{C}[x_1, \ldots, x_n] = \mathrm{Sym}(V)$

$f_1, \ldots, f_n$ homogeneous generators for A^W, $J = \det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j=1,\ldots,n} \neq 0$

algebra of differential forms: $A \otimes E$, $E = \bigwedge(V)$ = exterior algebra

= associative A-algebra generated by $dx_1, \ldots, dx_n$ w/ mult. $\wedge$

s.t. $dx_i \wedge dx_j = -dx_j \wedge dx_i$

$A \otimes E$ is free A-module of rank 2^n w/ basis dx_I

where for $I \subseteq \{1, \ldots, n\}$, $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_r}$

$A \otimes E$ is bigraded: if $h \in A$ is homogeneous, then $\deg(h\, dx_I) = (\deg h, |I|)$

If M is a bigraded vector space, let $M_{d,e}$ = (d,e)-bigraded piece, define

$$H_M(t,u) = \sum_{d,e \geq 0} \dim M_{d,e}\, t^d u^e.$$

$$H_{A\otimes E}(t,u) = H_A(t,u)\, H_E(t,u) = \frac{1}{(1-t)^n}(1+u)^n = \left(\frac{1+u}{1-t}\right)^n$$

$d(u+v) = du + dv$

W acts on $A \otimes E$: $g\left(\sum_I h_I\, dx_I\right) = \sum_I g h_I\, d(g x_{i_1}) \wedge \cdots \wedge d(g x_{i_r})$

Goal: Understand $(A \otimes E)^W$

Observation: given polynomials h_{ij} $i,j = 1, \ldots, n$:

$(*)$ $\left(\sum_i h_{i,1}\, dx_i\right) \wedge \left(\sum_i h_{i,2}\, dx_i\right) \wedge \cdots \wedge \left(\sum_i h_{i,n}\, dx_i\right) = \det(h_{ij})\, dx_1 \wedge \cdots \wedge dx_n$

$\Longrightarrow$ $g \in GL_n\mathbb{C}$, $g(dx_1 \wedge \cdots \wedge dx_n) = (\det g)\, dx_1 \wedge \cdots \wedge dx_n$

For any $h \in A$, define $dh = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i} dx_i$

Product rule: $d(h_1 h_2) = h_1 dh_2 + h_2 dh_1$

Lemma. $g \in GL_n\mathbb{C}$, then $g\,dh = d(gh)$

Pf. (1) True if $h = x_i$ by definition

(2) If true for h_1, h_2, true for product:

$$g\,d(h_1 h_2) = g(h_1 dh_2 + h_2 dh_1) = gh_1 \cdot d(gh_2) + h_2\, d(gh_1) = d(g(h_1 h_2))$$

(3) If true for h_1, h_2, true for $h_1 + h_2$

$\Rightarrow$ True for all h. $\square$

Def. $h \in A$ is skew-invariant if $gh = (\det g)^{-1} h$ for all $g \in W$.

Given reflection hyperplane H, let W_H be subgroup of W fixing H. Let $r_H = |W_H|$, ℓ_H = linear equation defining H.

Prop. (1) J is skew-invariant

(2) J = nonzero scalar multiple of $\prod_H \ell_H^{r_H - 1}$

(3) Every skew-invariant is of the form hJ for $h \in A^W$.

Pf. (1) Use $(*)$ w/ $h_{ij} = \frac{\partial f_j}{\partial x_i}$ to get

$$df_1 \wedge \dots \wedge df_n = J\, dx_1 \wedge \dots \wedge dx_n \text{ is } W\text{-invariant.}$$

Pick $g \in W$. $g(J\, dx_1 \wedge \dots \wedge dx_n) = gJ (\det g)\, dx_1 \wedge \dots \wedge dx_n$

$\Rightarrow gJ \det g = J \Rightarrow J$ skew-invariant.

(2) Let F be an arbitrary skew-invariant. Given H,

(4)(3)

Pick a basis $y_1, \dots, y_n$ for V: $y_1 = l_H$, $y_2 \dots y_n$ are fixed by W_H.

Pick generator $g \in W_H$. Then $g\, y_1^{p_1} \cdots y_n^{p_n} = (\det g)^{p_1} y_1^{p_1} \cdots y_n^{p_n}$

$g F = (\det g)^{r_H - 1} F \Rightarrow$ Since $\det g$ is primitive r_Hth root of unity,

F is divisible by $y_1^{r_H - 1} = l_H^{r_H - 1}$.

$\Rightarrow F$ is divisible by $\prod_H l_H^{r_H - 1}$.

$\deg J = \sum_{i=1}^{n} (d_i - 1) = \#$ of reflections $= \sum_H (r_H - 1)$

Since J is divisible by $\prod_H l_H^{r_H - 1}$ & same degree $\Rightarrow$ equal up to scalar multiple. $\square$

<u>Thm</u> (Solomon). $(A \otimes E)^W$ is freely generated as an exterior algebra over A^W by $df_1, \dots df_n$ i.e., every element of $(A \otimes E)^W$ is uniquely of the form $\sum_I h_I\, df_I$, $h_I \in A^W$, $df_I = df_{i_1} \wedge \dots \wedge df_{i_r}$.

<u>Pf.</u> <u>Uniqueness</u>: Suppose $\sum_I h_I\, df_I = 0$ where $h_I \in A^W$.

May assume all I in sum have same size. $I^c = [n] \setminus I$

Multiply by df_{I^c}: $df_I \wedge df_{I^c} = \pm J\, dx_1 \wedge \dots \wedge dx_n$

for $J \neq I$, $df_J \wedge df_{I^c} = 0$

$\Rightarrow \pm h_I J\, dx_1 \wedge \dots \wedge dx_n = 0 \Rightarrow \pm h_I J = 0 \Rightarrow h_I = 0$

<u>Existence</u>: Pick $\omega \in (A \otimes E)^W$. Suppose ω is homogeneous in the dx_i. Have $(A \otimes E)^W \subset A \otimes E \subset \mathrm{Frac}(A) \otimes E$

Argument of uniqueness proves df_I are linearly independent in $\mathrm{Frac}(A) \otimes E$: since $\mathrm{Frac} A \otimes E$ is 2^n-dim vectorspace / $\mathrm{Frac}(A)$

$\Rightarrow df_I$ are basis $\Rightarrow \exists\, c_I \in \mathrm{Frac} A$ s.t. $\omega = \sum_I c_I\, df_I$.

Average over W:

$$\omega = \frac{1}{|W|} \sum_I \left(\sum_{g \in W} g c_I \right) df_I = \sum_I \frac{s_I}{t_I} df_I$$

where $\frac{s_I}{t_I} = \sum_{g \in W} g c_I$, $s_I, t_I \in A$.

Multiply by df_{I^c}:

$$\pm c_I \partial \, dx_1 \wedge \dots \wedge dx_n = \underset{\substack{\cap \\ A \otimes E}}{\omega \wedge df_{I^c}} = \pm \frac{s_I}{t_I} \partial \, dx_1 \wedge \dots \wedge dx_n$$

$\Rightarrow$ $c_I \partial \in A$ & $c_I \partial \, dx_1 \wedge \dots \wedge dx_n$ is W-invariant.

$\Rightarrow$ $c_I \partial$ skew-invariant. $\Rightarrow$ $c_I \partial$ is divisible by ∂

$\Rightarrow$ $c_I \in A^W$. □

<u>Cor.</u> $H_{(A \otimes E)^W}(t,u) = \prod_{i=1}^n \frac{1 + t^{d_i - 1} u}{1 - t^{d_i}}$ where $d_i = \deg(f_i)$

<u>Pf.</u> Basis for $(A \otimes E)^W$ is $f_1^{p_1} \cdots f_n^{p_n} df_{i_1} \wedge \dots \wedge df_{i_r}$ where

$p_i \geq 0, 1 \leq i_1 < \dots < i_r \leq n$ □

Define $e_p(x_1, \dots x_n) = \sum_{1 \leq i_1 < \dots < i_p \leq n} x_{i_1} \cdots x_{i_p}$.

<u>Thm.</u> $e_p(d_1 - 1, \dots, d_n - 1) = \#\{g \in W \mid g \text{ has eigenvalue } 1 \text{ w/ multiplicity } n-p\}$

<u>Pf.</u> By Molien, $H_{(A \otimes E)^W}(t,u) = \frac{1}{|W|} \sum_{g \in W} \frac{\det(1 + gu)}{\det(1 - gt)}$

$$= \prod_{i=1}^n \frac{1 + t^{d_i - 1} u}{1 - t^{d_i}}$$

Compare coeffs of u^p: (for each $g \in W$, let $\omega_1(g), \ldots \omega_n(g)$ be its eigenvalues)

$$\frac{e_p(t^{d_1-1}, \ldots, t^{d_n-1})}{\prod_{i=1}^{n}(1-t^{d_i})} = \frac{1}{|W|}\sum_{g\in W} \frac{e_p(\omega_1(g), \ldots, \omega_n(g))}{(1-\omega_1(g)t)\cdots(1-\omega_n(g)t)}$$

Let $e_p'(x_1, \ldots, x_n) = e_p(1-x_1, \ldots, 1-x_n) =$ linear comb. of $e_0(x), e_1(x), \ldots, e_p(x)$

Can replace e_p above w/ e_p':

$$\frac{e_p(1-t^{d_1-1}, \ldots, 1-t^{d_n-1})}{\prod_{i=1}^{n}(1-t^{d_i})} = \frac{1}{|W|}\sum_{g\in W} \frac{e_p(1-\omega_1(g), \ldots, 1-\omega_n(g))}{(1-\omega_1(g)t)\cdots(1-\omega_n(g)t)}$$

Multiply by $(1-t)^{n-p}$ and evaluate at $t=1$. $\left([r] = \frac{1-t^r}{1-t}\right)$

LHS

$$\left.\frac{(1-t)^n e_p([d_1-1], \ldots, [d_n-1])}{\prod_{i=1}^{n}(1-t^{d_i})}\right|_{t=1} = \left.\frac{e_p([d_1-1], \ldots, [d_n-1])}{\prod_{i=1}^{n}[d_i]}\right|_{t=1}$$

$$= \frac{e_p(d_1-1, \ldots, d_n-1)}{d_1 \cdots d_n} = \frac{e_p(d_1-1, \ldots, d_n-1)}{|W|}$$

RHS

$$\frac{e_p(1-\omega_1(g), \ldots 1-\omega_n(g))}{(1-\omega_1 g t)\cdots(1-\omega_n g t)}$$

Case 1: 1 is eigenvalue w/ mult. $> n-p$

$\Rightarrow \quad e_p(1-\omega_1(g), \ldots 1-\omega_n(g)) = 0$

Case 2: 1 is eigenvalue w/ mult. $< n-p$

$$\left.\frac{(1-t)^{n-p}}{(1-\omega_1 t)\cdots(1-\omega_n t)}\right|_{t=1} = 0$$

Case 3. 1 is eigenvalue w/ mult. $= n-p$.

$$e_p(1-w_1 g, \dots 1-w_n g) = (1-w_1 g)\cdots(1-w_p g)$$ where $w_1 g, \dots, w_p g$ are eigenvalues $\neq 1$.

$$\frac{(1-t)^{n-p}}{(1-w_1 g t)\cdots(1-w_n g t)} = \frac{1}{(1-w_1 g t)\cdots(1-w_p g t)}$$

$\rightsquigarrow$ contribution is 1

$$RHS = \frac{1}{|W|} \# g \text{ where } 1 \text{ is eigenvalue w/ mult. } n-p$$ $\square$

Rmk. $p=1$: Thm says $e_1(d_1-1, \dots d_n-1) = \# g$ where 1 is eigenvalue w/ mult. $n-1$

$$e_1(d_1-1, \dots d_n-1) = \sum_{i=1}^{n}(d_i-1)$$

$$\# g \text{ w/ mult. } n-1 = \# \text{reflections}$$

$$d_1 \cdots d_n = ((d_1-1)+1)\cdots((d_n-1)+1) = \sum_{p=0}^{n} e_p(d_1-1, \dots, d_n-1) = |W|$$