Spherical Coxeter Complex

W = Coxeter group

V = geometric representation (real vector space)

$\mathbb{C}^n = V \otimes_{\mathbb{R}} \mathbb{C}$ complexification, $A = \mathbb{C}[x_1, \dots, x_n]$

$$D = \{ f \in V^* \mid f(\alpha_s) \geq 0 \quad \forall s \in S \}$$

Since B_W is nondegenerate, we have isomorphism $V \to V^*$, $x \to B_W(x, -)$

Can identify D w/ subset $\{ x \in V \mid B_W(x, \alpha_s) \geq 0 \ \forall s \in S \}$

The <u>fundamental chamber</u> for W acting on V

<u>Prop</u>. The union of W-translates of D is V.

<u>Pf</u>. Let $I \subseteq S$ be a subset. $\exists\, w_{I,0} \in W_I$ of maximal length. From before, $w_{I,0}(\alpha_s) < 0 \ \forall s \in I$, and $w_{I,0}(\alpha_s) = \alpha_s$ for $s \notin I$.

$$w_{I,0} D = \{ x \in V \mid B_W(x, \alpha_s) \leq 0 \text{ for all } s \in I,\ B_W(x, \alpha_s) \geq 0 \text{ for all } s \notin I \}.$$

The union of $w_{I,0} D$ over all subsets gives V. $\square$

The translates of D are <u>chambers</u>.

Each subset $I \subseteq S$ gives a face $D_I \subseteq D$ defined by

$$D_I = \{ x \in D \mid B_W(x, \alpha_s) = 0 \ \forall s \in I \}$$

$$S^{n-1} = \{ x \in V \mid B_W(x,x) = 1 \} \quad \text{(sphere of dim } n-1)$$

Intersect w/ set of translates of D_I. $\rightsquigarrow$ triangulation of S^{n-1}.

Spherical Coxeter complex

$\rightsquigarrow$ chain complex $F_\bullet$ which computes reduced homology of S^{n-1}. (w/ $\mathbb{Q}$-coefficients)

$\dim D_I = n - |I|$, $\quad \dim (S^{n-1} \cap D_I) = n - 1 - |I|$

F_i has basis given by i-dim'l faces = W-translates of D_I, $|I| = n-1-i$

$i = -1, \dots, n-1$

Reduced homology of S^{n-1} is given by

$$\tilde{H}_i(S^{n-1};\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } i=n-1 \\ 0 & \text{else} \end{cases}.$$

Prop. As class functions on W, we have

$$\det = \sum_{I\subseteq S} (-1)^{|I|} \mathbb{1} \uparrow_{W_I}^{W}$$

Pf. W acts on spherical Coxeter complex, so each $w\in W$ gives chain map $w: \mathcal{F}_\bullet \to \mathcal{F}_\bullet$

Hopf trace formula:

$$(-1)^{n-1} \operatorname{Tr}(w \mid \tilde{H}_{n-1}(S^{n-1};\mathbb{Q})) = \sum_{i=-1}^{n-1} (-1)^i \operatorname{Tr}(w \mid \mathcal{F}_i)$$

w is an orthogonal matrix, $\operatorname{Tr}(w \mid \tilde{H}_{n-1}(S^{n-1};\mathbb{Q})) = \det w$.

$\mathcal{F}_i$ = permutation rep. of W acting on set of W-translates of $\{D_I \mid |I| = n-1-i\}$

There is 1 orbit for every $I\subseteq S$ s.t. $|I|=n-1-i$ (reps are D_I)

and stabilizer of D_I is W_I.

$$\Longrightarrow \operatorname{Tr}(w\mid \mathcal{F}_i) = \sum_{\substack{I\subseteq S\\ |I|=n-1-i}} \mathbb{1}\uparrow_{W_I}^{W} \qquad \square$$

Recall. $W(t)$ (Poincaré series) is $\sum_{w\in W} t^{\ell(w)}$

$W \circlearrowright A = \mathbb{C}[x_1,\dots,x_n]$, $d_1,\dots,d_n$ = degrees of gens of A^W

$A^{W,\varepsilon}$ = module of skew-invariants of A.

We know $A^{W,\varepsilon}$ is a free A^W-module generated by J (Jacobian)

$$\deg(J) = \sum_{i=1}^{n}(d_i-1) = \#\text{ reflections of } W = \#\text{ positive roots} = \ell(w_0)$$

$$\Longrightarrow H_{A^{W,\varepsilon}}(t) = \frac{t^{\ell(w_0)}}{\prod_{i=1}^{n}(1-t^{d_i})}$$

Thm W = finite Coxeter group. Then (I = ideal gen. by A^W in A)

$$W(t) = H_{A/I}(t) = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}.$$

Pf. Let $Q_W(t) = H_{A/I}(t) = \prod_{i=1}^{n} \frac{1-t^{d_i}}{1-t}$.

Frobenius reciprocity: $\dim A_d^{W_I} = \langle 1\uparrow_{W_I}^{W}, \operatorname{char}(A_d)\rangle_W$

$$\sum_{I \subseteq S} (-1)^{|I|} H_{A^{W_I}}(t) = \sum_{I \subseteq S} (-1)^{|I|} \sum_{d \geq 0} \langle 1\uparrow_{W_I}^{W}, \operatorname{char}(A_d)\rangle_W \, t^d$$

$$= \sum_{d \geq 0} \langle \det, \operatorname{char}(A_d)\rangle_W \, t^d = H_{A^{W,\varepsilon}}(t)$$

$$= \frac{t^{\ell(w_0)}}{\prod_{i=1}^{n}(1-t^{d_i})} = \frac{t^{\ell(w_0)}}{(1-t)^n Q_W(t)}$$

If $|I| = p$, restriction of V to W_I is (geom. rep of W_I) $\oplus \mathbb{R}^{n-p}$.

$\Rightarrow A^{W_I}$ is gen by alg. ind. polys. of degrees $\underbrace{1,\dots,1}_{n-p}, \underbrace{d_1', \dots, d_p'}_{\text{degrees for } W_I}$

$$\Rightarrow H_{A^{W_I}}(t) = \frac{1}{(1-t)^{n-p}\prod_{i=1}^{p}(1-t^{d_i'})} = \frac{1}{(1-t)^n Q_{W_I}(t)}$$

Multiply first equation in proof by $(1-t)^n$: (and subtract)

$$\sum_{I \subsetneq S} (-1)^{|I|} \frac{1}{Q_{W_I}(t)} = \frac{t^{\ell(w_0)} - (-1)^{|S|}}{Q_W(t)}$$

Prove Thm by induction on $|S|$:

Base case: $|S| = 0$, $W(t) = 1 = Q_W(t)$ ✓

General:
$$\frac{t^{\ell(w_0)} - (-1)^{|S|}}{W(t)} = \sum_{I \subsetneq S} \frac{(-1)^{|I|}}{W_I(t)} = \sum_{I \subsetneq S} \frac{(-1)^{|I|}}{Q_{W_I}(t)} = \frac{t^{\ell(w_0)} - (-1)^{|S|}}{Q_W(t)}$$

$\Rightarrow W(t) = Q_W(t)$. $\square$

Rmk. $W = \mathfrak{S}_n$, $W(t) = \prod_{i=2}^{n} \frac{1-t^i}{1-t}$

invariants of $\mathfrak{S}_n$ acting on $\mathbb{C}[x_1, \dots, x_n]$ are degrees $1, 2, \dots, n$

When W is an affine Weyl group (Γ is positive semidefinite but not definite)

can get product formula for $W(t)$ using toroidal Coxeter complex