(W,S) Coxeter group. $V = \mathbb{R}\{\alpha_s \mid s\in S\}$, $B(\alpha_s,\alpha_t) = -\cos\left(\frac{\pi}{m(s,t)}\right)$

$\sigma_s(v) = v - 2B(\alpha_s, v)\alpha_s$ $\quad \sigma : W \longrightarrow GL(V)$, $s \longmapsto \sigma_s$ geometric representation of (W,S)

Def. length function ℓ on W is defined by $\ell(w)$ = minimum n s.t. $w = s_1 s_2 \cdots s_n$ where $s_i \in S$. A reduced expression for w is any expression $w = s_1 \cdots s_n$ where $n = \ell(w)$. $\ell(1) = 0$ by convention

Lemma. For all $w, w' \in W$, have:

1. $\ell(w) = \ell(w^{-1})$
2. $\ell(w) = 1 \iff w \in S$
3. $\ell(ww') \leq \ell(w) + \ell(w')$
4. $\ell(ww') \geq \ell(w) - \ell(w')$
5. For all $s\in S$, $\ell(ws), \ell(sw) \in \{\ell(w)-1, \ell(w)+1\}$

Pf. 1. If $w = s_1 \cdots s_n$ is reduced, so is $w^{-1} = s_n^{-1} \cdots s_1^{-1} = s_n \cdots s_1$

2. Clear

3. $w = s_1 \cdots s_n$ reduced, $w' = s_1' \cdots s_m'$ reduced

$ww' = s_1 \cdots s_n s_1' \cdots s_m' \implies n+m \geq \ell(ww')$.

4. By 3, $\underbrace{\ell(ww') + \ell(w'^{-1})}_{\ell(ww') + \ell(w')} \geq \ell(w)$, subtract $\ell(w')$.

5. $\ell(ws) \leq \ell(w) + \ell(s) = \ell(w)+1$ by 3

$\ell(ws) \geq \ell(w) - \ell(s) = \ell(w) - 1$ by 4

Note: if $\ell(w) = \ell(ws)$, then $\mathrm{sgn}(w) = (-1)^{\ell(w)} = (-1)^{\ell(ws)} = \mathrm{sgn}(ws) = -\mathrm{sgn}(w)$ $\times$.

$\implies \ell(ws) \in \{\ell(w)+1, \ell(w)-1\}$. Same for $\ell(sw)$. $\square$

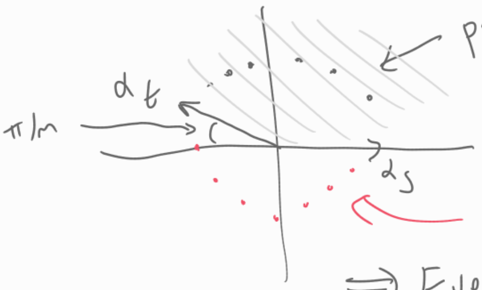
Define root system of W to be $\{w(\alpha_s) \mid s\in S, w\in W\}$. The elements are called roots. A root α is positive $(\alpha > 0)$ if α is a non-negative linear combination of $\{\alpha_s \mid s\in S\}$. A root α is negative $(\alpha<0)$ if α is a non-positive linear combination of $\{\alpha_s \mid s \in S\}$.

Ex. W = finite dihedral group, $S=\{s,t\}$, $m(s,t)=m<\infty$

The geometric representation of W is isometric to $\mathbb{R}^2$ w/ standard inner product so that $\alpha_s=(1,0)$, $\alpha_t=(\cos\theta,\sin\theta)$, $\theta=\pi-\frac{\pi}{m}$.

The roots are points $\left(\cos\left(\frac{j\pi}{m}\right),\sin\left(\frac{j\pi}{m}\right)\right)$, $0\le j<2m$, vertices of regular $2m$-gon.

$\sigma_s\sigma_t$ = rotation by $\frac{2\pi}{m}$



positive roots

$\underbrace{\cdots tst}_{\le m-1}$

negative roots (all -1 times a positive root)

$\Rightarrow$ Every root is either positive or negative.

Given subset $I\subseteq S$, define $W_I\subseteq W$ subgroup generated by I.

$\ell_I: W_I\to\mathbb{Z}_{\ge 0}$ defined by $\ell_I(w)$ = minimal n s.t. $w=s_1\cdots s_n$, $s_i\in I$.

Thm. Pick $w\in W$, $s\in S$. ① If $\ell(ws)>\ell(w)$, then $w(\alpha_s)>0$ (positive root)

② If $\ell(ws)<\ell(w)$, then $w(\alpha_s)$ is a negative root.

pf. ② follows from ① by using ws in place of w:

if $\ell(ws)<\ell(w)$, then $\ell(w)=\ell(wss)>\ell(ws)$ $\overset{①}{\Rightarrow}$ $\underset{w(-\alpha_s)}{\underset{\parallel}{ws(\alpha_s)}}>0$ $\Rightarrow$ $w(\alpha_s)<0$.

We prove ① by induction on $\ell(w)$. If $\ell(w)=0$, then $w=1$, so $\ell(s)>0$ $\forall s\in S$ & $1(\alpha_s)=\alpha_s>0$.

Now assume $\ell(w)>0$, pick $t\in S$ s.t. $\ell(wt)<\ell(w)$. So $t\ne s$, set $I=\{s,t\}$.

Define $A=\{(x,x_I)\in W\times W_I \mid w=xx_I,\ \ell(w)=\ell(x)+\ell_I(x_I)\}$.

Note: $(wt,t)\in A$, so $A\ne\emptyset$. Pick $(v,v_I)\in A$ s.t. $\ell(v)$ is minimal possible.

Then $\ell(v)\le\ell(wt)=\ell(w)-1$.

Claim: $\ell(vs)>\ell(v)$ & $\ell(vt)>\ell(v)$

Suppose not. Then
$$\begin{aligned}\ell(w)&\le\ell(vs)+\ell(sv_I)=(\ell(v)-1)+\ell(sv_I)\\&\le(\ell(v)-1)+\ell_I(sv_I)\\&\le(\ell(v)-1)+(1+\ell_I(v_I))=\ell(v)+\ell_I(v_I)=\ell(w)\end{aligned}$$

$\Rightarrow$ All $\leq$ are equalities. $\Rightarrow$ $l(w) = l(vs) + l_I(sv_I)$

$\Rightarrow$ $(vs, sv_I) \in A$. Since $l(vs) < l(v)$, this contradicts choice of v. $\square$

By induction, $v(\alpha_s) > 0$, $v(\alpha_t) > 0$. Suffices to prove $v_I(\alpha_s) > 0$ since $v_I(\alpha_s) \in \mathrm{Span}\{\alpha_s, \alpha_t\}$ & $w = v v_I$.

A reduced expression for v_I using s, t is an alternating product of s, t.
This must end in t (if not, then since $l(w) = l(v) + l_I(v_I)$, and $w = v v_I$ we get $l(ws) < l(w) \rightarrow\leftarrow$).

Let $m = m(s,t)$. <u>Case 1</u>: $m < \infty$. By explicit example.

<u>Case 2</u>. $m = \infty$. $v_I = (st)^k$ or $t(st)^k$, $k \geq 0$.

$$(\sigma_s \sigma_t)^k(\alpha_s) = (2k+1)\alpha_s + 2k\alpha_t > 0$$

$$\sigma_t(\sigma_s\sigma_t)^k(\alpha_s) = \sigma_t((2k+1)\alpha_s + 2k\alpha_t) = (2k+1)\alpha_s + (2k+2)\alpha_t > 0. \quad \square$$

<u>Cor</u>. Every root is either positive or negative

$$\underbrace{\Phi}_{\text{all roots}} = \underbrace{\Phi^+}_{\text{positive roots}} \amalg \underbrace{\Phi^-}_{\text{negative roots}}, \qquad \Phi^- = -\Phi^+.$$

<u>Pf</u>. $l(w) \neq l(ws)$ $\forall w \in W$, $s \in S$ $\square$

<u>Cor</u>. $\sigma: W \to GL(V)$ is injective.

<u>Pf</u>. Suppose $\sigma(w) = 1$, $w \neq 1$. $\exists$ $s \in S$ s.t. $l(ws) < l(w)$

By Thm $\alpha_s = w(\alpha_s) < 0$, contradiction. $\square$

<u>Ex</u>. A_{n-1} Coxeter system $W \twoheadrightarrow \mathfrak{S}_n$ = image of σ

$\Rightarrow$ $W \cong \mathfrak{S}_n$

$\Rightarrow$ $\mathfrak{S}_n \cong \left\langle s_1, \dots, s_{n-1} \;\middle|\; \begin{array}{l} s_i^2 = 1 \ \forall i \\ (s_i s_{i+1})^3 = 1 \ \forall i = 1, \dots, n-2 \end{array} \ (s_i s_j)^2 = 1 \text{ for } |i-j| > 1 \right\rangle.$