Affine representation of affine Weyl groups

(W_a, S_a) = Coxeter group whose graph is connected and positive semidefinite (but not pos. def).

V_a = geometric representation, $n+1 = \text{rank}(W_a, S_a)$

$\tilde{X}_n$ = name from classification

$s_0 \in S_a$ s.t. $(W, S_a \setminus s_0)$ is of type X_n

$\underset{\underset{V_a^\perp}{\parallel}}{\ker} B_{W_a}$ is 1-dim, spanned by δ s.t. coeff. of α_0 is 1

Define
$$Z = \{ f \in V_a^* \mid f(\delta) = 0 \}$$
$$E = \{ f \in V_a^* \mid f(\delta) = 1 \}$$

Z linear space, E affine space over Z

i.e., have simply transitive action $Z \times E \to E$

<u>Def.</u> $\varphi : E \to E$ is an <u>affine transformation</u> if $\exists$ linear map $\psi : Z \to Z$ s.t. $\varphi(e+z) = \varphi(e) + \psi(z)$ $\forall e \in E,\ z \in Z$

$\text{Aff}(E)$ = group of affine transformations under composition.

<u>Lemma.</u> $\text{Aff}(E) \cong \{ g \in GL(V_a^*) \mid g(E) = E \}$

<u>Pf.</u> Pick $g \in GL(V_a^*)$ s.t. $g(E) = E$. For $e \in E, z \in Z$, we have $g(e+z) = g(e) + g(z)$, so $g : E \to E$ is affine.

Conversely, suppose $\varphi : E \to E$ is affine, let $\psi : Z \to Z$ be corresponding linear map. Pick basis $v_0, \ldots, v_n$, s.t. $v_0 = \delta$,

for V_a. Let $v_0^*, \ldots, v_n^*$ be dual basis for V_a^*.
Then $v_1^*, \ldots v_n^*$ is basis for Z and we define $g \in GL(V_a^*)$

by $g(c_0 v_0^* + \cdots + c_n v_n^*) = c_0 \varphi(v_0^*) + \psi(c_1 v_1^* + \cdots + c_n v_n^*)$.

$$g\left((c_0 + c_0') v_0^* + \cdots + (c_n + c_n') v_n^*\right) = c_0 \varphi(v_0^*) + \psi(c_1 v_1^* + \cdots + c_n v_n^*) + c_0' \varphi(v_0^*) + \psi(c_1' v_1^* + \cdots + c_n' v_n^*)$$

$\Rightarrow$ linear

$g(E) = E$ (E corresponds to $c_0 = 1$) and $g|_E = \varphi$ since any element of E is of the form $v_0^* + v'$ where $v' \in \mathrm{Span}\{v_1^*, \ldots v_n^*\}$.
and so $g(v_0^* + v') = \varphi(v_0^*) + \psi(v') = \varphi(v_0^* + v')$. $\square$

Note. $w\delta = \delta$ $\forall w \in W_a$, so W_a preserves both Z, E

$\Rightarrow$ homomorphism $W_a \longrightarrow \mathrm{Aff}(E)$

This is injective.

B_{W_a} descends to pos. def. form on $V_a / V_a^\perp$ (call it B_W)
and we have $Z \cong (V_a/V_a^\perp)^*$. Using B_W, we can identify $(V_a/V_a^\perp) \cong (V_a/V_a^\perp)^*$

$$v_f \longleftarrow f$$

where v_f is unique vector s.t. $f(x) = B_W(v_f, x)$ $\forall x \in V_a/V_a^\perp$.

In particular, get W_a-invariant pos. def. form on Z.

For $s \in S_a$, define

$Z_s = \{ f \in V_a^* \mid f(\alpha_s) = 0 \}$

$E_s = E \cap Z_s$

Since coeff of α_0 in δ is 1, $\{\alpha_s \mid s \in S\} \cup \{\delta\}$ is linearly independent, so $E \cap \bigcap_{s \in S} Z_s$ is a single point e_0.

Explicitly, $e_0(\delta) = 1$, $e_0(\alpha_s) = 0$ for $s \neq s_0$

$$\Downarrow$$

$$e_0(\alpha_{s_0}) = 1$$

$\Rightarrow$ Identify $Z \cong E$ via $z \to z + e_0$.

Inverse denoted by $e' \to e' - e_0$ $\rightsquigarrow$ form B_E on E

<u>Lemma</u>. B_E is invariant under $(W_a)_{e_0}$.

<u>Pf</u>. Pick $w \in (W_a)_{e_0}$, so $we_0 = e_0$. Pick $e', e'' \in E$.

$$\begin{aligned} B_E(we', we'') &= B_W(we' - e_0, we'' - e_0) \\ &= B_W(w(e' - e_0), w(e'' - e_0)) \\ &= B_W(e' - e_0, e'' - e_0) \\ &= B_E(e', e''). \end{aligned}$$

$\square$

<u>Prop</u>. W_a is isomorphic to subgroup of $\mathrm{Aff}(E)$ which is generated by affine reflections.

<u>Pf</u>. Pick $s \in S$. E_s is an affine hyperplane passing through e_0. s fixes E_s, so fixes e_0, so preserves B_E.

Since $s^2 = 1$, it must be reflection w.r.t. E_s.

Now consider s_0. Let $\alpha_0' \in Z$ be the linear functional $v \to B_{W_a}(\alpha_{s_0}, v)$. Given $v \in V$, $z \in Z$,

$$(S_0(e_0+z))(v) = (e_0+z)(s_0 v)$$
$$= (e_0+z)(v - 2B_{W_a}(v, \alpha_{s_0})\alpha_{s_0})$$
$$= (e_0+z)(v) - 2(e_0+z)(\alpha_{s_0})\,\alpha_0'(v)$$
$$= (e_0+z)(v) - 2(1 + B_W(z, \alpha_0'))\,\alpha_0'(v)$$

$$\Rightarrow \quad S_0(e_0+z) = (e_0+z) - 2(1 + B_W(z, \alpha_0'))\alpha_0'$$

$\Rightarrow$ S_0 is an affine reflection $\square$