Semidirect product structure of affine Weyl groups

W = finite Weyl group of rank n

V = geometric representation of W

Φ = root system, $\alpha_1, \dots, \alpha_n$ simple roots, $(,)$ = form on V

L = $\mathbb{Z}$-span of $\alpha_1, \dots \alpha_n$, lattice preserved by W (root lattice)

$\tilde{\alpha}$ = highest root

Given $\alpha \in \Phi$, let $\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$, $L^\vee = \mathbb{Z}\text{-span}\{\alpha^\vee \mid \alpha \in \Phi\}$ <u>coroot lattice</u>

X_n = type of Φ, let Γ_a = Coxeter graph of type $\tilde{X}_n$

Γ_a obtained from Γ (= Coxeter graph of W) by adding node α_0

<u>Lemma</u>. Ker of bilinear form for Γ_a spanned by $\delta = \alpha_0 + \tilde{\alpha}$.

For $k \in \mathbb{Z}$, define $s_{\alpha,k} \in \mathrm{Aff}(V)$ by

$$s_{\alpha,k}(v) = v - ((\alpha, v) - k)\,\alpha^\vee$$

let $s_\alpha = s_{\alpha,0}$ (preserves form $(,)$). $s_{\alpha,k}$ is reflection w.r.t. affine hyperplane

$$H_{\alpha,k} = \{ v \in V \mid (\alpha, v) = k \}$$

For $v \in V$, let $t_v \in \mathrm{Aff}(V)$ denote translation $t_v(x) = x + v$.

<u>Lemma</u>. (a) If $w \in W$, then $wH_{\alpha,k} = H_{w\alpha,k}$ and $w s_{\alpha,k} w^{-1} = s_{w\alpha,k}$.

(b) $s_{\alpha,1} s_\alpha = t_{\alpha^\vee}$

(c) If $v \in V$, and $(v, \alpha) \in \mathbb{Z}$, then $t_v H_{\alpha,k} = H_{\alpha, k+(v,\alpha)}$ and $t_v s_\alpha t_v^{-1} = s_{\alpha, k+(v,\alpha)}$.

Pf. (a) Pick $x \in H_{\alpha,k}$. Then $(w\alpha, wx) = (\alpha, x) = k$,

so $w H_{\alpha,k} = H_{w\alpha,k}$.

Pick $x \in V$. Then

$$w s_{\alpha,k} w^{-1}(x) = w\left(w^{-1}x - ((\alpha, w^{-1}x) - k)\alpha^\vee\right)$$
$$= x - ((w\alpha, x) - k) w(\alpha^\vee) = s_{w\alpha,k}(x)$$

$w(\alpha^\vee) = w(\alpha)^\vee$

(b) Have $s_{\alpha,k} = t_{k\alpha^\vee} s_\alpha$. For $k=1$, $s_{\alpha,1} s_\alpha = t_{\alpha^\vee}$.

(c) If $x \in H_{\alpha,k}$, then $(\alpha, x+v) = k + (\alpha, v)$, so $x + v \in H_{\alpha, k+(\alpha,v)}$

For $x \in V$, $t_v s_{\alpha,k} t_v^{-1}(x) = s_{\alpha,k}(x - v) + v$

$$= (x - \cancel{v}) - ((\alpha, x - v) - k)\alpha^\vee + \cancel{v}$$
$$= x - ((\alpha, x) - (k + (\alpha, v)))\alpha^\vee$$
$$= s_{\alpha, k+(\alpha,v)}(x)$$

$\square$

We have $V \cong Z \cong E$

$$v \to B_{W_a}(v, -) \to B_{W_a}(v, -) + e_0$$

$\delta = \alpha_0 + \tilde{\alpha}$

Lemma. Via $V \cong E$, $s_{\tilde{\alpha},1}$ coincides w/ $s_0 \in W_a$.

Pf. We know s_0 is affine reflection on E, so need to compute its fixed hyperplane.

For $v \in V$, $z \in Z$, we have

$$(s_0(e_0+z))(v) = (e_0+z)(v - 2B_{W_a}(v,\alpha_0)\alpha_0)$$

Hence $s_0(e_0+z) = e_0+z \iff (e_0+z)(\alpha_0)=0$

Note: $e_0(\alpha_0)=1$ and $\alpha_0 \equiv -\tilde{\alpha} \pmod{V_a^{\perp}}$

$z(\tilde{\alpha})=1$

Hence for $u \in V$, under $V \cong E$, $z(\tilde{\alpha})=1$ translates to

$B_{W_a}(u,\tilde{\alpha})=1$, but B_{W_a} restricts to $(\,,\,)$ on V, so

the fixed hyperplane of s_0 is $H_{\tilde{\alpha},1}$. $\square$

<u>Cor.</u> W_a is generated by $s_{\alpha_1}, \dots s_{\alpha_n}, s_{\tilde{\alpha},1}$

<u>Cor.</u> (1) W_a contains t_v for all $v \in L^{\vee}$.

(2) For every $\alpha \in \Phi$, $k \in \mathbb{Z}$, we have $s_{\alpha,k} \in W_a$.

<u>Pf.</u> We just saw W_a is gen by $s_{\alpha_1}, \dots s_{\alpha_n}, s_{\tilde{\alpha},1}$.

Next, W acts transitively on Φ, so $s_{\alpha,1} \in W_a \;\forall \alpha \in \Phi$.

Since $s_{\alpha,1}s_{\alpha} = t_{\alpha^{\vee}}$, we also have $t_{\alpha^{\vee}} \in W_a \;\forall \alpha \in \Phi$.

$\Rightarrow t_v \in W_a$ for all $v \in L^{\vee} \Rightarrow$ (1)

(2) Note $(\alpha,\alpha^{\vee})=2$, so $t_{\alpha^{\vee}}^{k} s_{\alpha} t_{\alpha^{\vee}}^{-k} = s_{\alpha,2k}$

Since $s_{\alpha}, s_{\alpha,1} \in W_a \Rightarrow s_{\alpha,k} \in W_a \;\forall k \in \mathbb{Z}$. $\square$

$T_{L^{\vee}} = \{t_v \mid v \in L^{\vee}\}$, W acts on $T_{L^{\vee}}$ via

$w \cdot t_v = t_{wv}$.

Semidirect product $T_{L^\vee} \rtimes W$ is $T_{L^\vee} \times W$ as a set

w/ product $(t_v, w) \cdot (t_{v'}, w') = (t_v t_{wv'}, ww')$

Thm. $T_{L^\vee}$ is normal subgroup of W_a. $W_a \cong T_{L^\vee} \rtimes W$.

pf. $w t_v w^{-1} = t_{wv}$, so $T_{L^\vee}$ is normal subgroup.

Note: $W \cap T_{L^\vee} = \{id\}$ b/c W preserves 0 in V and only t_0 preserves v. $\Rightarrow$ $\varphi: W \longrightarrow W_a \longrightarrow W_a / T_{L^\vee}$ is injective. Next, $s_{\tilde{\alpha}} = s_{\tilde{\alpha},1} t_{\alpha^\vee}$ so $\varphi(s_{\tilde{\alpha}}) = s_{\tilde{\alpha},1} T_{L^\vee}$

and φ is also surjective, hence isomorphism.

$\Rightarrow$ get right coset representatives $\{T_{L^\vee} w \mid w \in W\}$

hence every element uniquely of the form $t_v w$ for $v \in L^\vee, w \in W$.

Finally, $w t_{v'} = t_{wv'} w$, so:

$t_v w t_{v'} w' = t_v t_{wv'} w w'$. $\Rightarrow$ $W_a \cong T_{L^\vee} \rtimes W$. $\square$