Affine permutation groups

<u>Type $\tilde{A}_{n-1}$:</u> $n \geq 2$ let $\tilde{\mathfrak{S}}_n$ = bijections $\sigma: \mathbb{Z} \to \mathbb{Z}$ s.t.
$\sigma(i+n) = \sigma(i) + n$ $\forall i \in \mathbb{Z}$ and s.t. $\sum_{i=1}^{n} \sigma(i) = \binom{n+1}{2}$

"window notation" $[\sigma(1), \ldots, \sigma(n)]$

$\Rightarrow$ σ induces bijection on $\mathbb{Z}/n$

<u>Lemma:</u> $\tilde{\mathfrak{S}}_n$ is a group under composition
↳ affine symmetric group

Let $L = \{x \in \mathbb{Z}^n \mid x_1 + \cdots + x_n = 0\}$, coroot lattice for $\mathfrak{S}_n$
For $x \in L$, let $t_x \in \tilde{\mathfrak{S}}_n$ be defined by $t_x(i) = i + x_i n$ for $i = 1, \ldots, n$

Extend notation so that x_i means $x_{i'}$ where $i' \in [n]$ is the coset representative for $i \in \mathbb{Z}/n$.
$\Rightarrow t_x(i) = i + x_i n \quad \forall i \in \mathbb{Z}$.
Then $t_x t_y = t_{x+y}$ and $t_x^{-1} = t_{-x}$. Let $T_L = \{t_x \mid x \in L\}$

We have surjection $\pi: \tilde{\mathfrak{S}}_n \to \mathfrak{S}_n$ by considering bijection mod n

$\sigma \in \ker \pi \iff \sigma(i) \equiv i \pmod n$ for $i = 1, \ldots, n$
$\iff \sigma(i) = i + x_i n$ for $x \in \mathbb{Z}^n$ s.t. $x_1 + \cdots + x_n = 0$
$\iff \sigma \in T_L$

$\Rightarrow \ker \pi = T_L$

We have injection $\mathfrak{S}_n \to \tilde{\mathfrak{S}}_n$ where for $\sigma \in \mathfrak{S}_n$, window notation is $[\sigma(1), \ldots, \sigma(n)]$

$\Rightarrow$ we have right coset representatives $\{T_L w \mid w \in \mathfrak{S}_n\}$ for $\tilde{\mathfrak{S}}_n / T_L$, so every element is uniquely of the form $t_x w$ for $x \in L$, $w \in \mathfrak{S}_n$.

Prop. We have isomorphism $W_a \xrightarrow{\cong} \tilde{S}_n$ where W_a is affine Weyl group of type $\tilde{A}_{n-1}$. s.t. $t_x w \to t_x w$.

Can make more explicit: for $i=1,\dots,n-1$,

$$s_i = [1,\dots,i+1,i,\dots n]$$

$$s_0 = [0,2,3,\dots,n-1,n+1]$$

Prop. For all $w \in \tilde{S}_n$, $\ell(w) = \sum_{1 \le i < j \le n} \left| \left\lfloor \frac{w(j)-w(i)}{n} \right\rfloor \right|$.

Type $\tilde{C}_n$: For $n \ge 2$, $N = 2n+1$.

$$\tilde{S}_n^C = \left\{ \sigma : \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \;\middle|\; \begin{array}{l} \sigma(i+N) = \sigma(i)+N \\ \sigma(i) = -\sigma(-i) \end{array} \right\}$$

$$\subseteq \tilde{S}_N$$

Let $L^\vee = \{(x_{-n},\dots,x_n) \in \mathbb{Z}^N \mid x_{-i} = -x_i\} \cong \mathbb{Z}^n$

For $x \in L^\vee$, define $t_x(i) = i + x_i N$ for $i=-n,\dots,n$.

Set $T_{L^\vee} = \{t_x \mid x \in L^\vee\} \subset \tilde{S}_n^C$

Let $W = W(B_n)$, thought of as permutations σ of $[-n,n]$ s.t. $\sigma(-i) = -\sigma(i)$

Get surjection $\pi : \tilde{S}_n^C \to W(B_n)$ by considering induced bijection modulo N (w/ reps $[-n,n]$)

As before, $\ker \pi = T_{L^\vee}$, and $\tilde{S}_n^C \cong T_{L^\vee} \rtimes W(B_n)$

Recall, type C_n root system consists of vectors $\pm e_i \pm e_j$ and $\pm 2e_i$ where $i,j \le n$ $\Rightarrow$ coroot lattice is $\mathbb{Z}^n$ w/ usual action of $W(B_n)$ by signed permutations.

Prop. $\widetilde{G}_n^C$ is affine Weyl group of type $\widetilde{C}_n$.

Type $\widetilde{B}_n$. Type B_n root system consists of vectors $\pm e_i \pm e_j$ and $\pm e_i$ for $i, j \le n$

Coroot lattice is $\{x \in \mathbb{Z}^n \mid x_1 + \cdots + x_n \text{ even}\}$ w/ induced action of W

For $\sigma \in \widetilde{G}_n^C$ and integers i, j define

$$\sigma[i,j] = \#\{a \in \mathbb{Z} \mid a \le i,\ \sigma(a) \ge j\}$$

Define $\widetilde{G}_n^B = \{\sigma \in \widetilde{G}_n^C \mid \sigma[n, n+1] \text{ is even}\}$.

Lemma. For $\sigma \in \widetilde{G}_n^C$, have

$$\sigma[n, n+1] = \sum_{i=1}^{n} \left\lfloor \frac{|\sigma(i)| + n}{N} \right\rfloor.$$

If $\sigma \in W(B_n)$, then $\sigma[n, n+1] = 0$, so $W(B_n) \subseteq \widetilde{G}_n^B$.

If $\sigma \in \widetilde{G}_n^C$ is t_x for $x = (x_{-n}, \ldots, x_n)$, then

$$\sigma[n, n+1] = x_1 + \cdots + x_n$$

Prop. $\widetilde{G}_n^B$ is affine Weyl group of $\widetilde{B}_n$.

Type $\widetilde{D}_n$. Coroot lattice for D_n is same as for B_n but $W(D_n)$ is subgroup of $W(B_n)$ s.t. # of negative signs is even.

Define $\widetilde{G}_n^D = \{\sigma \in \widetilde{G}_n^B \mid \sigma[0,1] \text{ even}\}$.

Lemma. For $\sigma \in \widetilde{G}_n^C$, $\sigma[0,1] = \sum_{i=1}^{n} \left\lfloor \frac{|\sigma(i)|}{N} \right\rfloor + \#\left\{ i \,\middle|\, \begin{array}{c} 1 \le i \le n \\ \sigma(i) < 0 \end{array} \right\}$

If $\sigma \in W(B_n)$ then $\sigma[0,1] = \#$ negative signs in $\sigma(1), \dots, \sigma(n)$

$\Longrightarrow \quad W(B_n) \cap \widetilde{G}_n^D = W(D_n)$

If $\sigma = t_x$, then $\sigma[0,1] = x_1 + \cdots + x_n$ (already even)

$(\sigma(i) = i + x_i N)$

Prop. $\widetilde{G}_n^D$ is affine Weyl group of type $\widetilde{D}_n$.