Hecke algebras

(W,S) = fixed Coxeter group.
A = commutative ring
$\mathcal{E}$ = free A-module w/ basis $\{T_w \mid w \in W\}$.

Thm Let $a, b \in A$. There is a unique associative A-algebra structure on $\mathcal{E}$ s.t. $\forall s \in S$, $w \in W$:

$$T_s T_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ aT_w + bT_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases}.$$

Denote this algebra by $\mathcal{E}(a,b)$.

Uniqueness is clear: $T_w = T_{s_1} T_{s_2} \cdots T_{s_r}$ if $w = s_1 s_2 \cdots s_r$ is a reduced expression. Hence, product $T_v T_w$ is determined by $T_s T_w$ in general.

Idea: construct this algebra in $\mathrm{End}(\mathcal{E})$.
For each $s \in S$, define $\lambda_s, \rho_s \in \mathrm{End}(\mathcal{E})$ by

$$\lambda_s(T_w) = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w) \\ aT_w + bT_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases}.$$

$$\rho_s(T_w) = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w) \\ aT_w + bT_{ws} & \text{if } \ell(ws) < \ell(w) \end{cases}.$$

Let $\mathcal{L}$ be subalgebra of $\mathrm{End}(\mathcal{E})$ generated by λ_s for $s \in S$.

Lemma: For all $s, t \in S$, have $\lambda_s \rho_t = \rho_t \lambda_s$.

In particular, ρ_t commutes w/ $\mathcal{L}$.

Lemma. Define $\varphi : \mathcal{L} \to \mathcal{E}$ by $\varphi(\lambda) = \lambda(T_1)$.
φ is an isomorphism of A-modules.

Pf. A-linearity is clear.
For $w \in W$, pick reduced expression $w = s_1 \cdots s_r$. Then

$$\varphi(T_{s_1} \cdots T_{s_r}) = T_{s_1} \cdots T_{s_r}(T_1) = T_w$$

$\Rightarrow \varphi$ is surjective.

Injectivity: Suppose $\varphi(\lambda) = 0$ for $\lambda \in \mathcal{L}$. WTS $\lambda = 0$.
Suffices to check that $\lambda(T_w) = 0 \ \forall w \in W$.
Prove this by induction on $\ell(w)$.
Base case: by assumption that $\varphi(\lambda) = \lambda(T_1) = 0$. ✓
Suppose $\ell(w) > 0$, write $w = vs$ for $s \in S$, $\ell(v) < \ell(w)$.
By induction, $\lambda(T_v) = 0$, so

$$\lambda(T_w) = \lambda(\rho_s T_v) = \rho_s \lambda(T_v) = 0 \qquad \square$$

For $w \in W$, define $\lambda_w = \lambda_{s_1} \cdots \lambda_{s_r}$ where $w = s_1 \cdots s_r$ is reduced expression. This is independent of choice of reduced expression.

Lemma. We have

$$\lambda_s \lambda_w = \begin{cases} \lambda_{sw} & \text{if } \ell(sw) > \ell(w) \\ a\lambda_w + b\lambda_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases}$$

Pf. $w = s_1 \cdots s_r$ reduced expression. If $\ell(sw) > \ell(w)$ then $ss_1 \cdots s_r$ is reduced, so $\lambda_{sw} = \lambda_s \lambda_w$.

Otherwise, $\ell(sw) < \ell(w)$. Using first case, have $\lambda_s \lambda_{sw} = \lambda_w$. Hence $\lambda_s \lambda_w = \lambda_s^2 \lambda_{sw}$, so suffices to show $\boxed{\lambda_s^2 = a\lambda_s + b}$.

Suffices to apply both to basis elements T_v.

① If $\ell(sv) > \ell(v)$:

$$\lambda_s^2(T_v) = \lambda_s(T_{sv}) = aT_{sv} + bT_v = (a\lambda_s + b)T_v \quad ✓$$

② If $\ell(sv) < \ell(v)$:

$$\lambda_s^2(T_v) = \lambda_s(aT_v + bT_{sv}) = a\lambda_s(T_v) + bT_v = (a\lambda_s + b)T_v \quad ✓$$

$\square$

$\Rightarrow$ $\varphi: \mathcal{L} \to \mathcal{E}$ is an isomorphism of A-modules & can transfer product structure on $\mathcal{L}$ to $\mathcal{E}$.

Let $A = \mathbb{Z}[q^{\pm 1/2}]$

Hecke algebra $\mathcal{H} = \mathcal{E}(q-1, q)$

Rmk. If we set $q=1$, get group algebra of W.