Parabolic subgroups $I \subseteq S$, W_I = subgroup of W generated by I.

$(I, m|_I)$ is Coxeter system $\rightsquigarrow$ get Coxeter group $\overline{W_I}$.

Get surjective homomorphism $\overline{W_I} \to W_I$

Lemma. $\overline{W_I} \to W_I$ is an isomorphism.

Pf. Let $V_I \subseteq V$ be subspace spanned by $\{\alpha_s \mid s \in I\}$.

V_I is naturally the geom. rep of $\overline{W_I}$; σ_I:

$$\begin{tikzcd} \overline{W_I} \arrow[rr] \arrow[dr] & & GL(V_I) \\ & W_I \arrow[ur] & \end{tikzcd}$$

σ_I is injective $\Rightarrow$ $\overline{W_I} \to W_I \to GL(V_I)$ also injective $\Rightarrow$ $\overline{W_I} \to W_I$ is injective. $\square$

W_I is a parabolic subgroup.

Thm. ① Pick $I \subseteq S$. If $w = s_1 \cdots s_r$ is reduced expression, $s_i \in S$, and $w \in W_I$, then $s_i \in I$ for all i. So $\ell_I = \ell|_{W_I}$.

② $I, J \subseteq S$, $W_I \cap W_J = W_{I \cap J}$. $W_{I \cup J}$ is subgroup generated by W_I & W_J.

③ S is a minimal generating set for W, i.e., no proper subset of S generates W.

Pf. ① Induction on $\ell(w)$. Base case: $\ell(w) = 0$, i.e., $w = 1$. Clear.

Assume now $\ell(w) > 0$. Set $s = s_r$. By last lecture, $w(\alpha_s) < 0$ since $\ell(ws) < \ell(w)$. Since $w \in W_I$, have $w = t_1 \cdots t_p$ w/ $t_i \in I$.

Hence $w(\alpha_s) = t_1 \cdots t_p(\alpha_s) = \alpha_s + \sum_{j=1}^{p} c_j \alpha_{t_j}$.

If $s \notin \{t_1, \dots, t_p\}$, then coeff of α_s in $w(\alpha_s)$ is 1 $\rightarrow\leftarrow$

$\Rightarrow$ $s \in I$ & $ws \in W_I$. By induction, since $ws = s_1 \cdots s_{r-1}$, we have $s_i \in I$ for $i = 1, \dots, r-1$. ✓

② Clear from ①

③ Suppose $I \subsetneq S$ generates W. Pick $s \notin I$. Then $s \in W_I = W$. By ①, $s \in I$. $\rightarrow\leftarrow$. $\square$

Lemma. $s \in S$, σ_s preserves $\Phi^+ \setminus \{\alpha_s\}$ & σ_s preserves $\Phi^- \setminus \{-\alpha_s\}$.

Pf. Pick $\alpha \in \Phi^+$, $\alpha \neq \alpha_s$. $B(\alpha,\alpha)=1 \Rightarrow \alpha$ not scalar multiple of α_s.

$\Rightarrow \alpha = \sum_{i \in S} c_i \alpha_i$ where $c_i \geq 0$ & $c_t > 0$ for some $t \neq s$.

$s(\alpha) = \alpha - 2B(\alpha, \alpha_s)\alpha_s \Rightarrow$ coeff of α_t in $s(\alpha)$ is $c_t > 0$.

$\Rightarrow s(\alpha)$ is positive root.

Also $s(\alpha) \neq \alpha_s$. If not, then $\alpha = s(\alpha_s) = -\alpha_s$ $\rightarrow\leftarrow$.

Second statement follows from first using that $\Phi^- = -\Phi^+$. □

Thm. For $w \in W$, $\ell(w) = |\{\alpha \in \Phi^+ \mid w(\alpha) < 0\}|$.

Pf. Induction on $\ell(w)$. Base case: $\ell(w)=0 \Rightarrow w=1$, clear.

So assume $\ell(w) > 0$, pick $s \in S$ s.t. $\ell(ws) = \ell(w)-1$. $\Rightarrow w(\alpha_s) < 0$.

$\Rightarrow ws(\alpha_s) = w(-\alpha_s) > 0$.

Pick $\alpha \in \Phi^+ \setminus \{\alpha_s\}$, then $s(\alpha) \in \Phi^+ \setminus \{\alpha_s\}$.

So $\{\alpha \in \Phi^+ \mid w(\alpha) < 0\} = \{\alpha_s\} \amalg \{\alpha \in \Phi^+ \mid ws(\alpha) < 0\}$

By induction, last set has size $\ell(ws) = \ell(w)-1 \Rightarrow |\{\alpha \in \Phi^+ \mid w(\alpha) < 0\}| = \ell(w)$. □

Ex. $W \cong \mathfrak{S}_n$, geom. rep. is (hyperplane in) $\mathbb{R}^n$, $w \in \mathfrak{S}_n$ acts by permuting coordinates. Roots are $e_i - e_j$ (up to scalar) for $i \neq j$.

Positive roots are $e_i - e_j$ $(i<j)$, negative roots are $e_i - e_j$ $(i>j)$.

For $i<j$, $e_i - e_j$ becomes negative $\Leftrightarrow w(i) > w(j)$.

$\Rightarrow \ell(w) = |\{ i<j \mid w(i) > w(j)\}|$ Such a pair is an inversion of w.

Given $\alpha = w(\alpha_s)$, define $s_\alpha = wsw^{-1}$. (Reflections)

Lemma. $\sigma(s_\alpha)(v) = v - 2B(v,\alpha)\alpha$. In particular, s_α does not depend on choice of w, s s.t. $\alpha = w(\alpha_s)$. Furthermore, $s_\alpha = s_{-\alpha}$, and for $\alpha, \beta \in \Phi^+$, $s_\alpha = s_\beta \Leftrightarrow \alpha = \beta$.

Pf. Write $\alpha = w(\alpha_s)$. Then

$$\sigma(S_\alpha)(v) = wsw^{-1}(v) = w\left(w^{-1}v - 2B(w^{-1}v, \alpha_s)\alpha_s\right)$$
$$= v - 2B(w^{-1}v, \alpha_s)\alpha = v - 2B(v,\alpha)\alpha.$$

σ injective $\Rightarrow$ s_α doesnot depend on w, s.

$s_\alpha = s_{-\alpha}$ clear from formula.

Now suppose $\alpha, \beta \in \Phi^+$, $s_\alpha = s_\beta$. Then

$$-\beta = s_\beta(\beta) = s_\alpha(\beta) = \beta - 2B(\beta,\alpha)\alpha \Rightarrow \beta = B(\beta,\alpha)\alpha$$

Both α, β are unit vectors, so $\beta = \pm\alpha$, both positive so $\beta = \alpha$. $\square$

T = set of all reflections, $T \cong \Phi^+$

= Union of conjugacy classes of $s \in S$.

Note: $\mathrm{sgn}(t) = -1$ for $t \in T \Rightarrow \ell(wt) \neq \ell(w)$ for all $w \in W, t \in T$

Ex. $W = \mathfrak{S}_n$, reflections are transpositions (i,j)

Thm. Pick $w \in W$, $t \in T$. Then $\ell(wt) > \ell(w) \Longleftrightarrow w(\alpha_t) > 0$.

Pf. First prove $\ell(wt) > \ell(w) \Rightarrow w(\alpha_t) > 0$ by induction on $\ell(w)$.

Base case clear. Assume $\ell(w) > 0$. Pick $s \in S$ s.th $\ell(sw) = \ell(w) - 1$.

Then $\ell(swt) \geq \ell(wt) - 1 > \ell(w) - 1 = \ell(sw)$

By induction, $sw(\alpha_t) > 0$. Suppose $w(\alpha_t) < 0$. Since s preserves $\Phi^- \setminus \{-\alpha_s\}$

$\Rightarrow w(\alpha_t) = -\alpha_s$. $\Rightarrow \alpha_t = w^{-1}s(\alpha_s) \Rightarrow t = (w^{-1}s)s(sw) = w^{-1}sw$.

$\Rightarrow wt = sw$. In particular, $\ell(wt) > \ell(w) > \ell(sw) \rightarrow\in$.

So $w(\alpha_t) > 0$.

Now need $\ell(wt) < \ell(w) \Rightarrow w(\alpha_t) < 0$.

Use previous part w/ wt in place of w: $\ell(wtt) > \ell(wt) \Rightarrow \underset{\displaystyle w(-\alpha_t)}{\underset{\shortparallel}{wt(\alpha_t)}} > 0$ $\square$