V = geometric representation of (W,S)

V^* = dual space of V, $f \in V^*$, $w \in W$, $v \in V$, $(wf)(v) = f(w^{-1}v)$.

For $s \in S$, define $Z_s = \{f \in V^* \mid f(\alpha_s) = 0\}$

$A_s = \{f \in V^* \mid f(\alpha_s) > 0\}$

$A_s' = \{f \in V^* \mid f(\alpha_s) < 0\}$

$C = \bigcap_{s \in S} A_s$ (positive orthant), $D = \overline{C} = \bigcap_{s \in S} \overline{A_s}$ (non-negative orthant)

$I \subseteq S$, $C_I = \bigcap_{s \in I} Z_s \cap \bigcap_{s \notin I} A_s = \{f \in D \mid f(\alpha_s) = 0 \Leftrightarrow s \in I\}$

$D = \coprod_{I \subseteq S} C_I$ partition of D

For $w \in W$, $w(D) = \{w(x) \mid x \in D\}$, $\mathcal{C} = \bigcup_{w \in W} w(D)$ is Tits cone.

Thm (a) For $w \in W$, $I, J \subseteq S$, if $w(C_I) \cap C_J \neq \emptyset$, then $I = J$ & $w \in W_I$.

(b) The stabilizer of any point of C_I is W_I.

(c) Every W-orbit on $\mathcal{C}$ intersects D in exactly one point.

Pf (a) Induction on $\ell(w)$. Base case $\ell(w) = 0 \Rightarrow w = 1$ clear.

Suppose $\ell(w) > 0$. Pick $s \in S$ s.t. $\ell(sw) < \ell(w)$. $\Rightarrow \ell(w^{-1}s) < \ell(w^{-1})$, so $w^{-1}(\alpha_s) < 0$. If $f \in D$, then $(wf)(\alpha_s) = f(w^{-1}\alpha_s) \leq 0$.

$\Rightarrow wf \in \overline{A_s'} \Rightarrow w(D) \subseteq \overline{A_s'}$.

$\Rightarrow w(C_I) \cap C_J \subseteq w(D) \cap D \subseteq \overline{A_s'} \cap \overline{A_s} = Z_s$

Pick $f \in w(C_I) \cap C_J$ (by assumption nonempty) $\Rightarrow f \in Z_s$, so $f(\alpha_s) = 0 \Rightarrow s \in J$.

If $g \in Z_s$, then $sg = g$: $sg(v) = g(v - 2B(\alpha_s, v)\alpha_s) = g(v)$.

$\Rightarrow s(C_J) = C_J$ & $f = sf \in \underbrace{sw(C_I) \cap C_J}_{\neq \emptyset}$.

By induction, $I = J$ & $sw \in W_I$. $\Rightarrow$ $w \in W_I$.

(b) Pick $x \in C_I$. Suppose $wx = x$. $\Rightarrow$ $w(C_I) \cap C_I \neq \emptyset$

$\overset{(a)}{\Rightarrow}$ $w \in W_I$.

If $w \in W_I$, then $w = s_1 \cdots s_n$, $s_i \in I$. Each s_i stabilizes C_I

$\Rightarrow$ w fixes every point of C_I.

(c) By definition every W-orbit in $\mathcal{C}$ intersects D. Pick $f, g \in D$ in same W-orbit, i.e., $f = w(g)$. $\exists I, J \subseteq S$ s.t. $f \in C_J$ & $g \in C_I$.

$\Rightarrow$ $w(C_I) \cap C_J \neq \emptyset$ $\overset{(a)}{\Rightarrow}$ $I = J$ & $w \in W_I$.

By (b) $f = wg = g$ $\square$

Pick basis for V, then can identify $GL(V) = GL_n(\mathbb{R})$ — subspace topology

$GL_n(\mathbb{R}) \subseteq \mathbb{R}^{n^2}$

$n \times n$ matrices $= \mathbb{R}^{n^2}$ — standard Euclidean topology

Given subset $A \subseteq GL(V)$, we say A is <u>discrete</u> if the subspace topology on A is discrete.

<u>Note</u>: $GL(V) \times V \longrightarrow V$ is continuous

$(g, v) \longrightarrow gv$

$GL(V) \times V^* \longrightarrow V^*$ also continuous

<u>Thm</u>. $W \subseteq GL(V)$ is discrete.

<u>Pf</u>. Pick $f \in C$, pick $w \in W$. Define $h: GL(V) \longrightarrow V^*$,

$g \longmapsto gw^{-1}f$

this is continuous. $C \subset V^*$ is open, so $h^{-1}(C) = C'$ also open.

If $v \in C'$, then $vw^{-1}f \in C$. If $v \in W$, then since $C = C_\phi$ $\Rightarrow vw^{-1} \in W_\phi \Rightarrow v = w$. $\square$