Representations of finite groups

G = finite group, identity element 1_G

A **linear representation** of G is a homomorphism $\rho: G \to GL(V)$,

V = complex vector space, $GL(V)$ = group of invertible linear operators $V \to V$

Equivalently, this is linear action of G on V: $g \cdot v$ s.t.

① $g\cdot(v+v') = g\cdot v + g\cdot v'$

② $(gg')\cdot v = g\cdot(g'\cdot v)$

③ $1_G \cdot v = v$

④ $g\cdot(\lambda v) = \lambda(g\cdot v) \quad \forall \lambda \in \mathbb{C}$

$$g\cdot v = \rho(g)(v)$$

Let V, V' be G-representations. A **G-equivariant map** $f: V \to V'$ is a linear map s.t. $f(g\cdot v) = g\cdot f(v) \; \forall v \in V, g \in G$

f is an isomorphism if it is so as a linear map: we write $V \cong V'$.

A subspace $W \subseteq V$ is a **subrepresentation** if $g\cdot w \in W \; \forall w \in W, g \in G$

V is **irreducible** if only subreps are 0 and V.

Ex. ① For any V, define $\rho(g) = id_V$

If $\dim V = 1$, this is the **trivial representation** of G.

② X = finite set w/ G-action. Can linearize it:

$\mathbb{C}[X]$ = vector space w/ basis $\{e_x \mid x \in X\}$. Define $g\cdot e_x = e_{g\cdot x}$

Permutation representation of G

Special case: $X = G$ w/ G acting by left multiplication.

$\mathbb{C}[G]$ = **regular representation** of G.

$\mathbb{C}[G]$ has multiplication: $e_g \cdot e_{g'} = e_{gg'}$. (group algebra)

A G-representation $\leftrightarrow$ left $\mathbb{C}[G]$-module: $\rho(g)\cdot v = e_g \cdot v$

Lemma ① All eigenvalues of $\rho(g)$ are roots of unity.

② $\rho(g)$ is diagonalizable.

Pf. ① $\rho(g)^N = 1$ for some $N \Rightarrow \lambda^N = 1$ for any eigenvalue λ

② Consider Jordan normal form: $\rho(g) \sim \begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$ + possibly some 1's

eigenvalues

1's $\Rightarrow$ $\rho(g)$ doesn't have finite order.

V = rep. of G, define $\mathrm{Tr}(g|V) = \mathrm{Tr}\,\rho(g)$

$\rightsquigarrow$ $\chi_V : G \to \mathbb{C}$ $\quad \chi_V(g) = \mathrm{Tr}(g|V)$

A <u>Class function</u> is a function $f : G \to \mathbb{C}$ invariant under conjugation, i.e., $f(g) = f(hgh^{-1})\ \forall g, h \in G$. χ_V is a class function

Inner product on class functions

complex conj.

$$\langle \varphi, \psi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)}\, \psi(g)$$

V, W = reps of G. <u>Basic operations</u>

• (Direct sum) $V \oplus W$ is rep: $g \cdot (v, w) = (g \cdot v, g \cdot w)$

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

• (Dual) V^* is a rep: $f \in V^*, g \in G, v \in V$

$$(gf)(v) = f(g^{-1}v), \quad \chi_{V^*} = \overline{\chi_V}$$

• (Tensor product) $V \otimes W$ is vector space spanned by symbols $v \otimes w$, $v \in V$, $w \in W$ s.t.

- $(v + v') \otimes w = v \otimes w + v' \otimes w$
- $v \otimes (w + w') = v \otimes w + v \otimes w'$
- $\lambda (v \otimes w) = (\lambda v) \otimes w = v \otimes \lambda w \quad \forall \lambda \in \mathbb{C}$

$V \otimes W$ is G-rep by $g \cdot \sum_i v_i \otimes w_i = \sum_i (g \cdot v_i) \otimes (g \cdot w_i)$

If $\{v_1, \dots v_n\}$, $\{w_1, \dots w_m\}$ bases for V, W, then $v_i \otimes w_j$ basis for $V \otimes W$

$\chi_{V \otimes W} = \chi_V \chi_W$.

- (Symmetric powers) $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$ has action of $\mathfrak{S}_n$ by permuting factors. $\mathrm{Sym}^n V = V^{\otimes n} \Big/ \left\langle \begin{array}{l} \sigma(x) = x \\ \text{for all } x \in V^{\otimes n} \\ \text{and all } \sigma \in \mathfrak{S}_n \end{array} \right\rangle$

  inherits G-rep structure.

  If $v_1, \dots v_r$ is a basis for V, then $\{v_{i_1} v_{i_2} \cdots v_{i_n} \mid i_1 \leq i_2 \leq \cdots \leq i_n\}$ is a basis for $\mathrm{Sym}^n V$

  where $v_{i_1} \cdots v_{i_n}$ is coset of $v_{i_1} \otimes \cdots \otimes v_{i_n}$.

- (Exterior powers) $\bigwedge^n V = V^{\otimes n} \Big/ \left\langle \begin{array}{l} \sigma(x) = \mathrm{sgn}(\sigma)\, x \\ \forall x \in V^{\otimes n}, \ \sigma \in \mathfrak{S}_n \end{array} \right\rangle$

  let $x_1 \wedge \cdots \wedge x_n$ be coset of $x_1 \otimes \cdots \otimes x_n$

  If $v_1, \dots v_r$ basis for V, then $\{v_{i_1} \wedge \cdots \wedge v_{i_n} \mid i_1 < i_2 < \cdots < i_n\}$ is a basis for $\bigwedge^n V$.

- (Invariants) $V^G = \{ v \in V \mid g \cdot v = v \ \forall g \in G \}$.

Prop. ① $\chi_V = \chi_{V'} \iff V \cong V'$

② If $\chi_V(g) = 0 \ \forall g \neq 1$, then $V \cong \mathbb{C}[G]^{\oplus N}$ where $N = \frac{\dim V}{|G|}$.

③ χ_V is real-valued $\iff V \cong V^*$

④ $\langle \chi_V, \chi_{V'} \rangle_G = \dim (V^* \otimes V')^G$

$H \subseteq G$ subgroup. Let V be G-rep. By restricting domain, V is also H-representation, called <u>restriction</u> $\mathrm{Res}^G_H V$

Let V be an H-representation. $\Rightarrow$ V is a left $\mathbb{C}[H]$-module.

$$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V = \mathrm{Ind}^G_H V.$$

(let R be a ring, M be right R-module, N be left R-module

$M \otimes_R N$ = abelian group spanned by $m \otimes n$, $m \in M$, $n \in N$ s.t.

- $(m+m') \otimes n = m \otimes n + m' \otimes n$
- $m \otimes (n+n') = m \otimes n + m \otimes n'$
- $mr \otimes n = m \otimes rn \quad \forall r \in R$)

$R = \mathbb{C}[H]$, $M = \mathbb{C}[G]$ right action: $e_g \cdot e_h = e_{gh}$
$N = V$

$\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$ is a G-rep by $g(\sum e_{g_i} \otimes v_i) = \sum e_{gg_i} \otimes v_i$

$$\dim \mathrm{Ind}^G_H V = (\dim V)\frac{|G|}{|H|}$$

<u>Ex</u>. X = set w/ transitive G-action. i.e., $\forall x, y \in X$ $\exists g \in G$ s.t. $gx = y$

Pick point $x \in X$, let $H \subseteq G$ be its stabilizer. Then

$$G/H \xrightarrow{\sim} X \qquad \text{isomorphism of } G\text{-sets}$$
$$gH \longrightarrow g \cdot x \qquad \text{where } g \cdot (g'H) = gg'H$$

So $\mathbb{C}[G/H] \cong \mathbb{C}[X]$. Furthermore, $\mathbb{C}[G/H] \cong \mathrm{Ind}^G_H \mathbb{C}$
$$e_{gH} \to e_g \otimes 1$$

The character of $\mathbb{C}[G/H]$ is denoted $1 \uparrow^G_H$

Thm (Frobenius reciprocity) $H \subseteq G$, $U = H$-rep. $V = G$-rep.

Then $\langle \chi_{\mathrm{Ind}_H^G U}, \chi_V \rangle_G = \langle \chi_U, \mathrm{Res}_H^G \chi_V \rangle_H$

Cor. $\langle 1\uparrow_H^G, \chi_V \rangle_G = \dim V^H$