Classification of finite Coxeter groups

Choose ordering on S $\rightsquigarrow$ symmetric matrix $(A_W)_{ij} = B_W(\alpha_i, \alpha_j) = -\cos\left(\frac{\pi}{m(i,j)}\right)$

- $(A_W)_{ij} \leq 0$ if $i \neq j$
- $(A_W)_{ii} = 1$

Given $S' \subseteq S$, det of submatrix of A_W whose rows/cols indexed by S' is a principal minor.

Recall: A_W is positive definite if all principal minors are positive.
A_W is positive semidefinite ——— ≥ 0

Pos def $\iff$ $B_W(v,v) > 0$ for all $v \neq 0$
$\iff$ $v^T A_W v > 0$ for all $v \neq 0$

Pos semidef $\iff$ $B_W(v,v) \geq 0$ for all v $\iff$ $v^T A_W v \geq 0$ $\forall v$

Def. A $n\times n$ symmetric matrix A is decomposable if $\exists$ subset $S \subseteq [n]$ s.t. $A_{ij} = 0$ $\forall i \in S$, $j \notin S$. A is indecomposable if not decomposable.

A_W is indecomposable $\iff$ Γ is connected.

Thm. (Perron–Frobenius). A = real $n\times n$ symmetric matrix, positive semidefinite, indecomposable, $A_{ij} \leq 0$ $\forall i \neq j$. Then

(a) Define $N = \{x \in \mathbb{R}^n \mid x^T A x = 0\}$. Then $N = \ker A$ & $\dim N \leq 1$.

(b) The smallest eigenvalue has multiplicity 1, and has eigenvector whose coordinates are all positive.

Pf. A real symmetric $\Rightarrow$ $A = P^T D P$, P orthogonal, D is diagonal w/ real entries $d_1 \geq d_2 \geq \cdots \geq d_n$. A pos. semidef. $\Rightarrow$

$$0 \leq (P^{-1} e_n)^T A (P^{-1} e_n) = e_n^T D e_n = d_n.$$

(a) If $x \in N$, set $y = Px$. $\Rightarrow \sum_{i=1}^{n} d_i y_i^2 = 0$. Since $d_i \geq 0$

$\Rightarrow \forall i$ either $d_i = 0$ or $y_i = 0 \Rightarrow \sum_{i=1}^{n} d_i y_i = 0$

$\Rightarrow DPx = 0 \Rightarrow x \in \ker A \Rightarrow N \subseteq \ker A \Rightarrow N = \ker A$.

Suppose $N \neq 0$, pick nonzero $x \in N$. Let $z = \sum_{i=1}^{n} |x_i| e_i$.

$$0 \leq z^T A z = \sum_{i=1}^{n} A_{ii} x_i^2 + \sum_{i \neq j} A_{ij} |x_i x_j|$$

$=$

$$\leq \sum_{i=1}^{n} A_{ii} x_i^2 + \sum_{i \neq j} A_{ij} x_i x_j = x^T A x = 0$$

$\Rightarrow z \in N$. Let $J = \{j \mid z_j \neq 0\}$. Claim: $J = [n]$

Suppose not, pick $i \notin J$. Since $N = \ker A$, we have $Az = 0$

$\Rightarrow 0 = (Az)_i = \sum_{j \in J} \underbrace{A_{ij}}_{\leq 0} \underbrace{z_j}_{> 0} \Rightarrow A_{ij} = 0 \ \forall j \in J, i \notin J$

$\rightarrow\leftarrow$ (A indecomposable)

Suppose $x, y \in N$ nonzero. (all coordinates are nonzero)

$N \ni x - \frac{x_1}{y_1} y$ has 0 coordinate $\Rightarrow x = \frac{x_1}{y_1} y \Rightarrow \dim N \leq 1$.

(b) If $d_n = 0$, we're done.

If $d_n \neq 0$, consider $A - d_n I$. Its kernel is eigenspace for eigenvalue d_n for A. $\square$

We'll call Γ_w <u>positive (semi) definite</u> if A_w has this property.

Γ' is a <u>subgraph</u> of Γ if obtained by deleting vertices or edges.

Proper subgraph means $\neq \Gamma$.

<u>Cor</u>. Γ = connected, positive semidef. graph. Every proper subgraph is positive definite.

Pf. Let A be symmetric matrix of Γ.

Suppose $\exists$ subgraph Γ' s.t. A' (sym. matrix of Γ') is not pos. def.

Relabel vertices so that Γ' uses $1, \dots, k$ of the n vertices.

$\exists\, x \in \mathbb{R}^n$ s.t. $x \neq 0$ & $x^T A' x \leq 0$. Let $y = (|x_1|, \dots, |x_k|, 0, \dots 0)^T$.

$-\cos$ is increasing on $[0, \frac{\pi}{2}] \Rightarrow A'_{ij} \geq A_{ij}\ \forall i,j$.

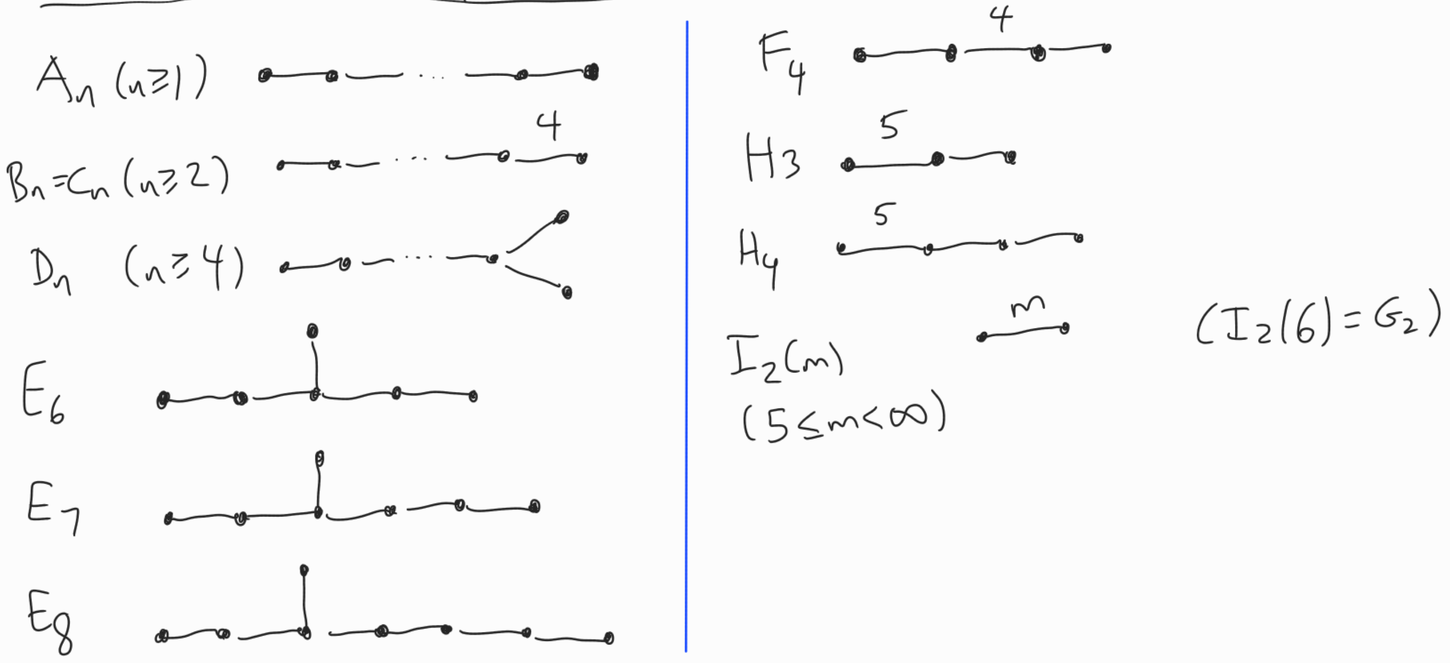
$$0 \underset{=}{\leq} y^T A y = \sum_{i,j} A_{ij} |x_i x_j| \underset{=}{\leq} \sum_{i,j} A'_{ij} |x_i x_j| \leq \sum_{i,j} A'_{ij} x_i x_j = x^T A' x \leq 0$$

$y \in N$ (from previous result) $\Rightarrow$ all coordinates of y are positive

$\Rightarrow k = n$

$A_{ij} = A'_{ij}\ \forall i,j \Rightarrow \Gamma' = \Gamma$. $\square$

Positive Definite graphs (finite Coxeter groups)

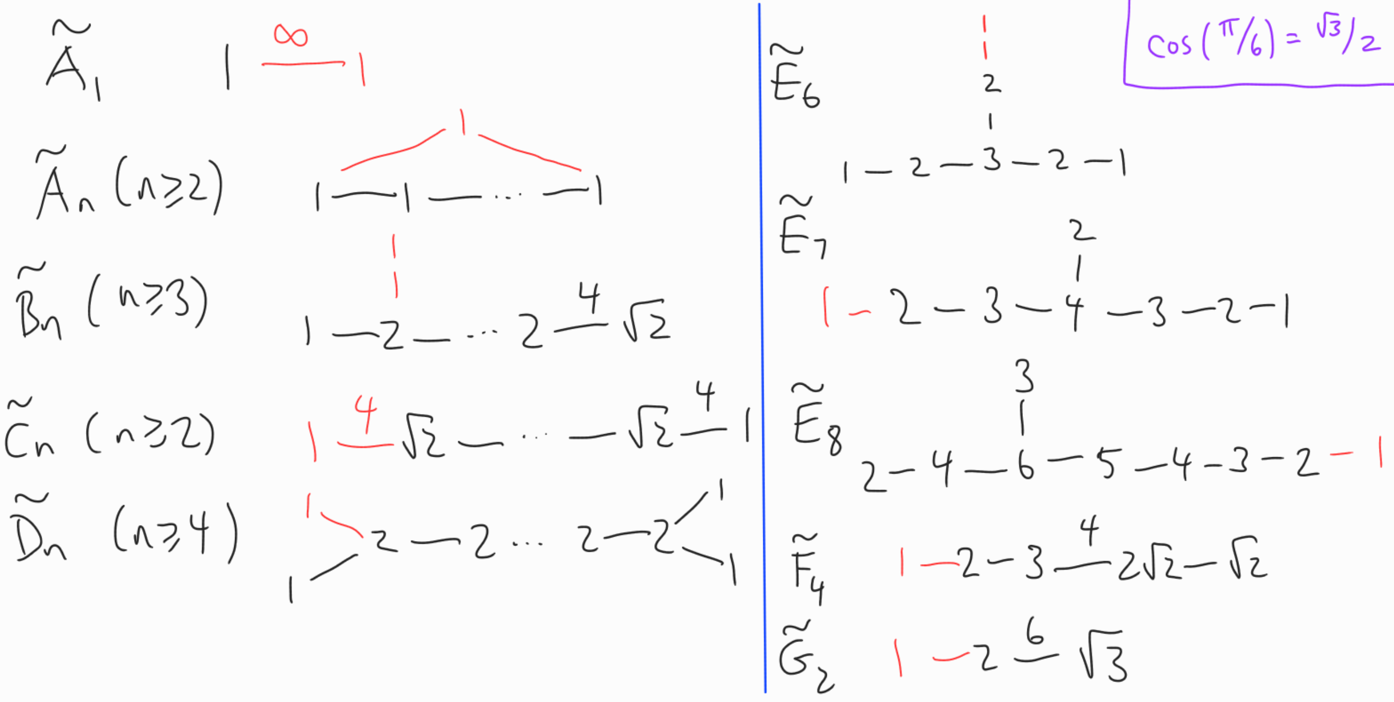


Positive semidefinite graphs (affine Coxeter groups)

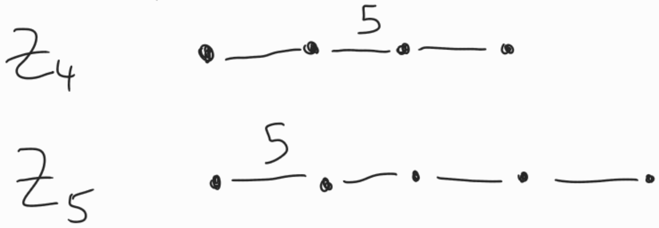
$\cos(\pi/3) = 1/2$

$\cos(\pi/4) = 1/\sqrt{2}$

$\cos(\pi/6) = \sqrt{3}/2$



Not positive semidefinite:



$\cos(\pi/5) = \frac{1+\sqrt{5}}{4}$

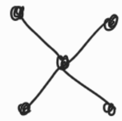
Thm. Every positive semidef, connected Coxeter graph appears in one of the previous 2 lists.

Pf. Suppose not. Let Γ be pos. semidef. connected not on these lists.
$n = \#$vertices, $m = $ maximum label of an edge.

- $n \geq 3$, $m < \infty$, Γ has no cycles.

Case 1. $m = 3$

- m has some vertex of degree ≥ 3
  This vertex is unique, and has degree $= 3$ (no $\tilde{D}_4$)

$\tilde{D}_4$: 

$a \leq b \leq c$

(no $\tilde{E}_6 \Rightarrow a = 1$) (no $\tilde{E}_7 \Rightarrow b \leq 2$) ($\neq D_n \Rightarrow b = 2$)
(no $\tilde{E}_8 \Rightarrow c \leq 4$) ($\neq E_6, E_7, E_8 \rightarrow c \neq 2,3,4 \rightarrow \leftarrow$)

Case 2. $m \geq 4$
(no $\tilde{C}_n \Rightarrow$ exactly one edge has label ≥ 4)
(no $\tilde{B}_n \Rightarrow$ no vertices w/ degree > 2)

Case 2a. $m = 4$
($\neq B_n \Rightarrow$ unique edge w/ label 4 has to be in middle $\Rightarrow n \geq 4$)
(no $\tilde{F}_4 \Rightarrow n \leq 4$)
($\neq F_4 \Rightarrow n \neq 4$) $\rightarrow\leftarrow$

Case 2b $m \geq 5$
(no $\tilde{G}_2 \Rightarrow m = 5$)
(no $Z_4, Z_5 \Rightarrow n \leq 4$ & edge w/ label 5 is at end)
($\neq H_3, H_4 \Rightarrow n \neq 3, n \neq 4 \rightarrow \leftarrow$)

$\square$