Math 188, Fall 2022
Homework 1
Due: October 8, 2022 11:59PM via Gradescope
(late submissions allowed up until October 9, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Find a closed formula for the following recurrence relation:

$$a_0 = 1,\ a_1 = 1,\ a_2 = 2,$$
$$a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \qquad (n \geq 3).$$

You may use a computer for factoring or solving a system of equations if you like, just say so in your submission. You should still write out all other steps though.

(2) Let $r_1, \dots, r_d$ be distinct numbers. Show that the determinant of the $d \times d$ matrix $(r_i^{j-1})_{i,j=1,\dots,d}$ is nonzero (interpret $0^0 = 1$). Explain why this implies that the sequences $(r_1^n)_{n\geq 0}, \dots, (r_d^n)_{n\geq 0}$ are linearly independent.

[Hint: If you're having trouble, look up "Vandermonde determinant". There are a lot of ways to prove that the determinant is nonzero, find your favorite way and rewrite it in your own words. If you do this, cite your source.]

(3) Let $(a_n)_{n\geq 0}$ be a sequence satisfying a linear recurrence relation whose characteristic polynomial is $(t^2 - 1)^d$.

Show that there exist polynomials $p(n)$ and $q(n)$ of degree $\leq d-1$ such that

$$a_n = \begin{cases} p(n) & \text{if } n \text{ is even} \\ q(n) & \text{if } n \text{ is odd} \end{cases}.$$

(4) (a) Suppose that $(a_n)_{n\geq 0}$ and $(a'_n)_{n\geq 0}$ both satisfy the same linear recurrence relation of order d and that they agree in d consecutive places, i.e., there exists k such that $a_k = a'_k, a_{k+1} = a'_{k+1}, \dots, a_{k+d-1} = a'_{k+d-1}$. Show that these sequences are the same.

(b) Suppose that $(a_n)_{n\geq 0}$ satisfies the linear recurrence relation of order d

$$a_n = c_1 a_{n-1} + \cdots + c_d a_{n-d} \qquad \text{for all } n \geq d$$

with $c_d \neq 0$. Show that there is a unique sequence $(b_n)_{n\in\mathbf{Z}}$ (indexed by *all* integers) such that $b_n = a_n$ for $n \geq 0$ and such that

$$b_n = c_1 b_{n-1} + \cdots + c_d b_{n-d} \qquad \text{for all } n \in \mathbf{Z}.$$

(c) Consider the Fibonacci sequence $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$. How does the negatively indexed Fibonacci sequence relate to the usual one?

(5) Let $A_0(x), A_1(x), \dots$ and $B_0(x), B_1(x), \dots$ be sequences of formal power series. Assume that $\lim_{i\to\infty} A_i(x) = A(x)$ and $\lim_{i\to\infty} B_i(x) = B(x)$.

(a) Prove that $\lim_{i\to\infty} (A_i(x) + B_i(x)) = A(x) + B(x)$.

(b) Prove that $\lim_{i\to\infty} (A_i(x)B_i(x)) = A(x)B(x)$.

Optional problems (don't turn in)

(6) Continuing from (3), how does the statement generalize if the characteristic polynomial is $(t^k - 1)^d$?

(7) Let p be a prime number and let $(a_n)_{n\geq 0}$ be a sequence such that $a_n \in \mathbf{Z}/p$ and which satisfies a homogeneous linear recurrence relation. Show that the sequence is in fact periodic.

(8) Let $r_1, \dots, r_{d-1}$ be distinct numbers. Prove that the sequences $\alpha_1 = (r_1^n), \dots, \alpha_{d-1} = (r_{d-1}^n), \alpha_d = (nr_{d-1}^{n-1})$ are linearly independent by showing that the determinant of $(\alpha_{i,j-1})_{i,j=1,\dots,d}$ is nonzero (interpret $0^0 = 1$ and if $r_{d-1} = 0$, interpret $\alpha_{d,0} = 0$).