Math 188, Fall 2022
Homework 2
Due: October 15, 2022 11:59PM via Gradescope
(late submissions allowed up until October 16, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let $F(x)$ be a formal power series with $F(0) = 0$.
   (a) Show that there exists a formal power series $G(x)$ with $G(0) = 0$ such that $F(G(x)) = x$ if and only if $[x^1]F(x) \neq 0$.
   (b) Assuming $[x^1]F(x) \neq 0$, show that $G(x)$ is unique and also satisfies $G(F(x)) = x$. You may use without proof that composition of formal power series is associative.

(2) Evaluate the following sums:
   (a) $\displaystyle\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i}$
   (b) $\displaystyle\sum_{i=0}^{n} i^2 \binom{n}{i} 3^i$

(3) Let a, b be non-negative integers.
   (a) By comparing coefficients in $(1+x)^{a+b} = (1+x)^a(1+x)^b$, prove that for any non-negative integer n, we have

$$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}.$$

   (b) Now prove this identity using a counting argument.
   [Hint: Consider choosing n animals from a dogs and b cats...]

(4) How many ways can we arrange the letters of: MISSISSIPPI ?

(5) Let $f(t) = \sum_{k=0}^{d} f_k t^k$ be a degree d polynomial with rational coefficients. From lecture, we know that there exist unique rational numbers $g_0, \dots, g_d$ such that

$$\sum_{n \geq 0} f(n) x^n = \frac{g_0 + g_1 x + \cdots + g_d x^d}{(1-x)^{d+1}}.$$

Now assume that $f(a)$ is an integer for $a = 0, \dots, d$. (The f_k don't have to be integers for this to be true, for example $f(n) = n(n-1)/2$ has this property.)

Prove that this implies that the g_k are all integers and that $f(a)$ is an integer whenever a is an integer.

[Hint: first prove that

$$f(t) = \sum_{k=0}^{d} g_k \binom{d+t-k}{d},$$

as an identity of polynomials in t, and then consider the system of equations you get from $t = 0, \dots, d$.]

Optional problems (don't turn in)

(6) Let $n \geq 2$ be an integer.

(a) Prove that
$$\sum_{i=0}^{n} i\binom{n}{i}(-1)^{i-1} = 0.$$

(b) Compute
$$\sum_{\substack{0\leq i\leq n\\ i \text{ even}}} i\binom{n}{i}.$$

(7) (a) Let a, b be rational numbers. Show that for any formal power series $A(x)$ with $A(0) = 1$, we have
$$A(x)^a A(x)^b = A(x)^{a+b}.$$
[Remember that we defined rational powers in a very specific way, so your proof needs to use this definition.]

(b) Deduce from (a) that
$$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}$$
for all non-negative integers n.

(8) Assume now that we deal with complex-coefficient formal power series. Define the following sets of formal power series:
$$V = \{F(x) \mid F(0) = 0\},$$
$$W = \{G(x) \mid G(0) = 1\}.$$

(a) Given $F \in V$, show that $\mathbf{E}(F) = \sum_{n\geq 0} \frac{F(x)^n}{n!}$ is the *unique* formal power series $G \in W$ such that $DG = DF \cdot G$. This defines a function $\mathbf{E}\colon V \to W$. [Convention: $F(x)^0 = 1$ even if $F(x) = 0$.]

(b) Given $G \in W$, show that there is a *unique* formal power series $F \in V$ such that $DF(x) = DG(x)/G(x)$. We define the function $\mathbf{L}\colon W \to V$ by $\mathbf{L}(G) = F$. [For the rest, it is unnecessary to use explicit formulas for $\mathbf{L}$ and $\mathbf{E}$ and in fact it may be easier to only use the uniqueness properties above.]

(c) Show that $\mathbf{E}$ and $\mathbf{L}$ are inverses of each other.

(d) Show that $\mathbf{E}(F_1 + F_2) = \mathbf{E}(F_1)\mathbf{E}(F_2)$ for all $F_1, F_2 \in V$.

(e) Show that $\mathbf{L}(G_1G_2) = \mathbf{L}(G_1) + \mathbf{L}(G_2)$ for all $G_1, G_2 \in W$.

(f) If m is a positive integer and $G \in W$, show that $\mathbf{E}(\frac{\mathbf{L}G}{m})$ is an mth root of G. [This gives an alternative proof for the existence of mth roots and in fact we can now define powers for any complex number m: $F^m = \mathbf{E}(m\mathbf{L}(F))$.

(g) Show that if $\sum_{i\geq 0} F_i(x)$ converges to $F(x)$, then $\prod_{i\geq 0} \mathbf{E}(F_i)$ converges to $\mathbf{E}(F)$.

(h) Show that if $\prod_{i\geq 0} G_i(x)$ converges to $G(x)$, then $\sum_{i\geq 0} \mathbf{L}(G_i)$ converges to $\mathbf{L}(G)$.