Math 188, Fall 2022
Homework 4
Due: November 5, 2022 11:59PM via Gradescope
(late submissions allowed up until November 6, 2022 11:59PM with −25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) (a) Let r be a fixed nonnegative integer. Show that both $S(n+r,n)$ and $c(n+r,n)$ are polynomial functions of n of degree $2r$ for $n \geq 0$.
   (b) Compute these polynomials for $r = 2, 3$.

(2) For $n > 0$, let a_n be the number of partitions of n such that every part appears at most twice, and let b_n be the number of partitions of n such that no part is divisible by 3. Set $a_0 = b_0 = 1$. Show that $a_n = b_n$ for all n.

(3) Let y be a variable. Prove the following generalization of Example 3.27

$$\prod_{i\geq 0}(1+x^{2i+1}y) = \sum_{r\geq 0} \frac{x^{r^2}y^r}{(1-x^2)(1-x^4)\cdots(1-x^{2r})}.$$

Hint on next page.

(4) (a) Use the following q-analogue of Pascal's identity (you don't need to prove it)

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \qquad \text{(for } n \geq k > 0\text{)}$$

to show that if d is a non-negative integer, then

$$\sum_{n\geq 0} \begin{bmatrix} n+d \\ n \end{bmatrix}_q x^n = \prod_{i=0}^{d}(1-q^i x)^{-1} = \frac{1}{(1-x)(1-qx)\cdots(1-q^d x)}.$$

   (b) Give a direct explanation (i.e., independent of the Schubert decomposition explanation from lecture) for why the coefficient of x^n of the right side is the sum $\sum_\lambda q^{|\lambda|}$ over all integer partitions λ whose Young diagram fits in the $n \times d$ rectangle.

(5) Let V, W be $\mathbf{F}_q$-vector spaces with $\dim V = n$ and $\dim W = m$.
   (a) How many linear maps $V \to W$ are there?
   (b) Suppose $n \geq m$. How many surjective linear maps $V \to W$ are there?
   (c) Pick $k \leq \min(m,n)$. How many rank k linear maps $V \to W$ are there?

See next page for some hints.

Hints

3: If you expand this out, the coefficient of x^n is a polynomial in y. If we set $y = 1$, this is just Example 3.27 and the coefficient of x^n counts some special kind of partitions of n on each side which are in bijection. By examining that proof more carefully, figure out what the coefficient of $y^k x^n$ means on each side and show that these descriptions match up under the bijection.

5a: Picking bases for V and W, you can represent a linear map uniquely as a matrix.

5c: A rank k linear map is essentially the same thing as a surjective linear map onto some k-dimensional subspace of W.

Optional problems (don't turn in)

(6) This one is challenging (I can't see a solution that is simple, but it doesn't require anything outside of this class). Prove:

$$\sum_{n \geq 1} x^{n(n-1)/2} = \prod_{n \geq 1} \frac{1 - x^{2n}}{1 - x^{2n-1}}.$$

(7) Pick integers satisfying $1 \leq k_1 < \cdots < k_r \leq n$. Let X be the set of subspaces $W_1, \dots, W_r$ of $\mathbf{F}_q^n$ such that $\dim W_i = k_i$ for all i and $W_i \subset W_{i+1}$ for $i < r$.

- (a) Find a formula for $|X|$ using by generalizing Example 3.39, i.e., use a q-analogue of a multinomial coefficient.
- (b) $|X|$ is also a polynomial in q; find an explicit description of this polynomial using a generalization of the Schubert decomposition of the Grassmannian.