Math 190A, Fall 2022
Homework 1
Due: October 7, 2022 11:59PM via Gradescope
(late submissions allowed up until October 8, 2022 11:59PM with −25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) (a) Let X be a set and let I be an index set. Suppose that for each $i \in I$, we have a topology $\mathcal{T}_i$ on X.
Prove that $\mathcal{T} = \bigcap_{i \in I} \mathcal{T}_i$ is also a topology on X.
Prove that $\mathcal{T} \leq \mathcal{T}_i$ for all $i \in I$, and in fact, if there is another topology $\mathcal{T}'$ such that $\mathcal{T}' \leq \mathcal{T}_i$ for all $i \in I$, then $\mathcal{T}' \leq \mathcal{T}$ (i.e., $\mathcal{T}$ is the "greatest lower bound" of all of the $\mathcal{T}_i$).
(b) Let $X = \{1, 2, 3\}$ and find two topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on X such that $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology.

(2) Let X be a topological space with topology $\mathcal{T}$, let A be a subset of X, and let B be a subset of A, i.e., $B \subseteq A \subseteq X$. There are two potentially different topologies we can put on B: First, B is a subset of X so we can give it the subspace topology $\mathcal{T}_B$. Second, we can give A the subspace topology $\mathcal{T}_A$ from X, and then give B the subspace topology $(\mathcal{T}_A)_B$ that comes from being a subset of A.
Prove that they are actually the same: $\mathcal{T}_B = (\mathcal{T}_A)_B$.

(3) Let X be a topological space and let A be a subspace. Prove that if U is open in A, then for any other subset B of X, $U \cap B$ is open in the subspace $A \cap B$.

(4) Let X be a topological space and let A, B be subsets of X.
(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
(b) Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
(c) Give an example where $\overline{A} \cap \overline{B}$ is not equal to $\overline{A \cap B}$.
[Hint: There is an example where $X = \mathbf{R}$ and A, B are open intervals.]

(5) Let X be a topological space and let A be a subset of X. Prove the identities

$$X \setminus \overline{A} = (X \setminus A)^\circ, \qquad X \setminus A^\circ = \overline{X \setminus A}.$$

Optional problems (don't turn in)

(6) Let I be an index set and suppose we have a topological space X_i for each $i \in I$. Let X be the disjoint union of all of the X_i:

$$X = \coprod_{i \in I} X_i.$$

Formally, this is the set of pairs $\{(i, x) \mid i \in I,\ x \in X_i\}$. Let $\mathcal{T}$ be the collection of subsets U of X such that for all $i \in I$, the set $U_i = \{x \in X_i \mid (i, x) \in U\}$ is open in X_i. Prove that $\mathcal{T}$ is a topology for X.

(7) Let $X = \mathbf{Z}$ be the set of integers. For each pair of integers m, n such that $m \neq 0$, define the subset

$$b_{m,n} = \{mx + n \mid x \in \mathbf{Z}\}.$$

- (a) Prove that the collection of $b_{m,n}$ (with $m \neq 0$ but no restriction on n) form a basis for a topology, which we will just call $\mathcal{T}$.
  [Remark: Since each $b_{m,n}$ is infinite, all non-empty open sets in $\mathcal{T}$ are infinite.]
- (b) Prove that each $b_{m,n}$ is both open and closed in $\mathcal{T}$.
- (c) Prove that
  $$\mathbf{Z} \setminus \{1, -1\} = \bigcup_p b_{p,0}$$
  where the union is over all prime numbers p.
- (d) Using the above facts, conclude that there must be infinitely many primes.
  [Hint: use proof by contradiction.]

(8)
- (a) Let X be a topological space. Given a subset A, define $f(A) = \overline{A}$, so that we have a function $f\colon 2^X \to 2^X$ which we call closure. Prove that f satisfies these 4 properties:
  - (i) $f(\varnothing) = \varnothing$.
  - (ii) For all $A \subseteq X$, we have $A \subseteq f(A)$.
  - (iii) For all $A \subseteq X$, we have $f(A) = f(f(A))$.
  - (iv) For all $A, B \subseteq X$, we have $f(A) \cup f(B) = f(A \cup B)$.
- (b) Conversely, suppose that Y is a set and we are given a function $g\colon 2^Y \to 2^Y$ satisying the 4 conditions above. Prove that there is a unique topology on Y so that g is the closure function of this topology.
  In particular, this says that we could define topologies in terms of functions satisfying (i)–(iv) instead of with open sets.
- (c) Find and prove the analogous statement for the function that takes a subset to its interior.