Math 190A, Fall 2022
Homework 3
Due: October 28, 2022 11:59PM via Gradescope
(late submissions allowed up until October 29, 2022 11:59PM with -25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f\colon X \to Y$ be a function such that
$$d_Y(f(a), f(b)) = d_X(a, b)$$
for all $a, b \in X$. Prove that f is an embedding.

(2) Let X be a topological space and define $\Delta_X \subseteq X \times X$ by
$$\Delta_X = \{(x, x) \mid x \in X\}.$$
Prove that X is Hausdorff if and only if Δ_X is closed.

(3) Let I be an index set and X_i a topological space for each $i \in I$. Let $x(1), x(2), \ldots$ be a sequence of elements in $\prod_{i \in I} X_i$. Given $x \in \prod_{i \in I} X_i$, prove that $x(1), x(2), \ldots$ converges to x if and only if for all $j \in I$, the sequence $x(1)_j, x(2)_j, \ldots$ converges to x_j.

(4) Let $f\colon X \to Y$ be a surjective continuous function and assume that for all open sets $U \subseteq X$, $f(U)$ is also open. Define an equivalence relation $\sim$ on X by $x \sim y$ if $f(x) = f(y)$. Prove that $X/{\sim} \cong Y$.

(5)
  (a) Finish the missing detail in Example 2.4.7 from the notes: given the function $f\colon [0,1] \to \mathrm{S}^1$ defined by $f(x) = (\cos(2\pi x), \sin(2\pi x))$, prove that for $0 < a < b < 1$, the images $f((a, b))$ and $f([0, a) \cup (b, 1])$ are open in S^1.
  (b) Define an equivalence relation $\sim$ on $\mathbf{R}$ by $x \sim y$ if $x - y$ is an integer. Prove that $\mathbf{R}/{\sim} \cong \mathrm{S}^1$.

Optional problems (don't turn in)

(6) Let d be a metric on a set X. Prove that $d\colon X \times X \to \mathbf{R}_{\geq 0}$ is continuous, where X has the metric topology.

(7) Let d be a metric on a set X. Define $d'\colon X \times X \to \mathbf{R}_{\geq 0}$ by

$$d'(x,y) = \frac{d(x,y)}{1 + d(x,y)}.$$

Prove that d' is a metric and that d and d' give X the same metric topology.

[Hint: Define $f\colon \mathbf{R}_{\geq 0} \to \mathbf{R}$ by $f(x) = x/(1+x)$. To show that d' satisfies the triangle inequality, first prove that f is increasing and that for all $a, b \in \mathbf{R}_{\geq 0}$, we have $f(a) + f(b) \geq f(a+b)$.]

[Remark: d' is *bounded* since the distance between any two points is bounded from above by a constant (in this case 1). This exercise shows that every metrizable space always comes from a bounded metric. In particular, a metric being bounded has no interesting implication on the resulting topology!]