Math 190A, Fall 2022
Homework 6
Due: Friday, December 2, 2022 11:59PM via Gradescope
(late submissions allowed up until December 3, 2022 11:59PM with −25% penalty)

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Let $X$ and $Y$ be spaces and set $\{p\} = X^* \setminus X$ and $\{q\} = Y^* \setminus Y$. Prove that $(X \amalg Y)^* \cong (X^* \amalg Y^*)/\sim$ where the only nontrivial relation is $p \sim q$.

[Recall that $\amalg$ means "disjoint union", and if $A$ and $B$ are spaces, then a subset $U$ of $A \amalg B$ is open if and only if $U \cap A$ and $U \cap B$ are open in $A$ and $B$, respectively.]

(2) Let $X = \mathbf{Z}_{>0}$ be the set of positive integers with the discrete topology.

(a) Prove that $X$ is locally compact, Hausdorff, and not compact.

(b) Prove that $X^*$ is homeomorphic to the subspace $\{0\} \cup \{1/d \mid d \in \mathbf{Z}_{>0}\}$ of $\mathbf{R}$.

(3) Let $Y$ be a Hausdorff space and let $X \subseteq Y$ be a locally compact subspace such that $\overline{X} = Y$. Prove that $X$ is an open subset of $Y$. Hints at end.

(4) Let $n \geq 1$ be an integer. Recall that $\mathbf{RP}^n = (\mathbf{R}^{n+1} \setminus \{\mathbf{0}\})/\sim$ where $x \sim y$ if there exists $\lambda \in \mathbf{R} \setminus \{0\}$ such that $x = \lambda y$. Write $[x_1 : \cdots : x_{n+1}]$ for the equivalence class of $(x_1, \ldots, x_{n+1})$. Our goal is to show that $\mathbf{RP}^n$ is a compactification of $\mathbf{R}^n$; since it's a bit lengthy this problem will be worth 40 points rather than the usual 20.

Define $U \subseteq \mathbf{RP}^n$ to be the subset of equivalence classes of the form $[x_1 : \cdots : x_{n+1}]$ where $x_{n+1} \neq 0$ (this makes sense since whether or not $x_{n+1}$ is 0 does not depend on the actual representative).

(a) Prove that the function $g \colon U \to \mathbf{R}^n$ given by

$$g([x_1 : \cdots : x_{n+1}]) = \left(\frac{x_1}{x_{n+1}}, \ldots, \frac{x_n}{x_{n+1}}\right)$$

is well-defined (i.e., does not depend on the choice of representative for the equivalence class) and is a homeomorphism. Hint at end.

(b) Prove that $\overline{U} = \mathbf{RP}^n$.

(c) Finally, prove that $\mathbf{RP}^n$ is Hausdorff.

[You may use that the restriction $\pi|_{\mathrm{S}^n} \colon \mathrm{S}^n \to \mathbf{RP}^n$ is a quotient map, i.e., $U \subseteq \mathbf{RP}^n$ is open if and only if $(\pi|_{\mathrm{S}^n})^{-1}(U)$ is open. This does not follow from definitions and requires a proof, but you can take it for granted for this problem.]

(d) When $n = 1$, explain why $\mathbf{RP}^1 \setminus U$ is a single point and explain how this implies that $\mathbf{RP}^1 \cong \mathrm{S}^1$.

## Hints

3: Hint 1: By Proposition 4.3.19 (taking $U = X$), each $x \in X$ has a neighborhood $V \subseteq X$ which is open in $X$ such that $\mathrm{Cl}_X(V)$ is compact. Prove that $V$ is also open in $Y$ (see next hint for more help).

Hint 2: Continuing from hint 1, $V = X \cap W$ for some open set $W$ in $Y$. Explain why each of the following equalities holds:

$$W \subseteq \mathrm{Cl}_Y(W) = \mathrm{Cl}_Y(V) = \mathrm{Cl}_X(V) \subseteq X.$$

4a: To show that $g$ is continuous: let $\widetilde{U} = \pi^{-1}(U)$ where $\pi\colon \mathbf{R}^{n+1} \setminus \{\mathbf{0}\} \to \mathbf{RP}^n$ is the quotient map. Define $f\colon \widetilde{U} \to \mathbf{R}^n$ by the same formula as $g$ and show that $f(x) = g(\pi(x))$ for all $x \in \widetilde{U}$. Now use an argument very similar to the proof of Proposition 2.4.5.

## Optional problems (don't turn in)

(5) How do you describe $(X \times Y)^*$ in terms of $X^*$ and $Y^*$?

(6) Prove that $\mathbf{CP}^n$ is a compactification of $\mathbf{C}^n$ and that $\mathbf{CP}^1 \cong \mathrm{S}^2$.

(7) Pick $0 < k < n$. Prove that $\mathbf{Gr}_k(\mathbf{R}^n)$ is a compactification of $\mathbf{R}^n$ and that $\mathbf{Gr}_k(\mathbf{C}^n)$ is a compactification of $\mathbf{C}^n$.