Math 190A, Fall 2022
Homework 7
This is for practice only, don't turn it in. But there's a very high chance that one of these problems will appear on the final exam.

(1) Let X be a compact metrizable space. Prove that X is second-countable.

(2) Let X be a regular space. Prove that for all $x, y \in X$ with $x \neq y$, there exist neighborhoods U and V of x and y, respectively, such that $\overline{U} \cap \overline{V} = \varnothing$.

(3) A space X is called **completely regular** if it is a T_1-space, and given any closed subset A and $x \notin A$, there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$.
   (a) Let X be a locally compact and Hausdorff space. Prove that X is completely regular. [Hint: Use X^*.]
   (b) Prove that completely regular implies regular.
   (c) Use the previous two parts to conclude that every manifold is metrizable.

(4) Let X be a compact and Hausdorff space and assume that for each $x \in X$, there is a neighborhood U of x and a positive integer k such that U has an embedding into $\mathbf{R}^k$. Prove that X has an embedding into $\mathbf{R}^N$ for some N.

   [Note: this is close to the definition of locally Euclidean, but we're allowing k to depend on x, and not fixing it ahead of time.]