NOTES FOR MATH 190A

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1. Definitions

1.1. **Set theory preliminaries.** A lot of this course is going to involve working with operations on sets, so let's set up some common notation and conventions.

First, we need to address indexing objects by a set I. This just means that for each element $i \in I$, we have some object, usually a set, or a number, that we might denote with a subscript, like U_i or x_i, etc. You're probably used to the situation where I is a set of integers like $1, 2, \dots,$ but we need to be more flexible and can't always use this. We'll denote the objects by notation like $\{U_i\}_{i \in I}$.

Usually I has some natural meaning to our context. For example, suppose $I = \mathbf{R}$ is the set of real numbers and U_i is the interval of length 2 centered at i: $U_i = [i-1, i+1]$.

Now let's focus on the case that we have a fixed set X and our indexed objects are subsets U_i of X. Then we can define the union and intersection of all of the subsets in the usual way and we denote them by

$$\bigcup_{i \in I} U_i, \qquad \bigcap_{i \in I} U_i.$$

If I is empty, then the standard conventions are that the union of 0 subsets is the empty set, while the intersection of 0 subsets is all of X.

Here's a useful identity: let I, J be index sets and let $\{U_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ be subsets of X indexed by I and J, respectively. Then

$$(\bigcup_{i \in I} U_i) \cap (\bigcup_{j \in J} V_j) = \bigcup_{i \in I} \bigcup_{j \in J} (U_i \cap V_j)$$

(exercise: prove it).

1.2. **What is a topological space?** The basic objects of study in this course are topological spaces and continuous functions.

What "counts" as a topological space has changed over time and while there is now a standard definition (which we will give soon), it is important to note here that even now how we think about or define topological spaces can change based on context. Simply put, while we have a fairly general and elegant definition, it is sometimes "too general" and there are really pathological examples that we may want to eliminate. A standard way is to define properties and only consider spaces which satisfy those properties, but other approaches involve giving different definitions altogether (we may touch on this towards the end).

The definition we will use is via "open sets". You have likely encountered this idea in calculus or real analysis via a δ-ϵ definition. Intuitively, open sets are supposed to be subsets which have "wiggle room".

Given a set X, let 2^X denote its power set, i.e., the set of all subsets of X.

Definition 1.2.1. Let X be a set. A **topology** on X is a subset $\mathcal{T} \subseteq 2^X$ satisfying the following properties:

(1) $\varnothing \in \mathcal{T}$.
(2) $X \in \mathcal{T}$.
(3) (Finite intersections): For any *finite* subset $\{U_1, \dots, U_n\} \subseteq \mathcal{T}$, their intersection also belongs to $\mathcal{T}$:

$$U_1 \cap U_2 \cap \cdots \cap U_n \in \mathcal{T}.$$

(4) (Arbitrary unions): For any index set I and collection $\{U_i\}_{i\in I}$ where $U_i \in \mathcal{T}$, their union is also in $\mathcal{T}$:
$$\bigcup_{i\in I} U_i \in \mathcal{T}.$$ □

For (4), it's important to keep in mind that I does not have to be finite (or even countable). We really have to allow for all possible index sets.

Definition 1.2.2. A **topological space** is a pair $(X, \mathcal{T})$ where $\mathcal{T}$ is a topology on X. Usually we'll just write "X is a topological space" with the unspoken assumption that the information $\mathcal{T}$ is also there to avoid having clunky notation.

If $\mathcal{T}$ is a topology on X, then any element of $\mathcal{T}$ is called an **open subset** of X. □

In other words a topology tells us which sets are called "open". We can rephrase the definition as saying that we have a topology if and only if the following properties are true:

(1) The empty set is open.
(2) The whole set X is open.
(3) Any finite intersection of open sets is again open.
(4) Any union (including infinite ones) of open sets is again open.

Remark 1.2.3. Using induction, we can replace condition (3) by only requiring that the intersection of any two open subsets is open (but you cannot do this for (4)!). □

Example 1.2.4. Let X be any set. There are always 2 obvious topologies we can put on X:

(1) We can take $\mathcal{T} = 2^X$. Then *every* subset is open and it is clear that the conditions hold. This is called the **discrete topology** on X.
(2) We can take $\mathcal{T} = \{\varnothing, X\}$. This is the bare minimum we could possibly do and it works also. This is the **indiscrete topology** on X. □

Example 1.2.5. Let $X = \mathbf{R}$ be the set of real numbers. We will call a subset U open if it has the following property: for any $x \in U$, there exists $\epsilon > 0$ such that the interval $(x - \epsilon, x + \epsilon)$ is contained in U. I won't verify the conditions that these define a topology on $\mathbf{R}$ (you should try it yourself) since in any case this will be a special case of something we discuss soon. Note that $U = \varnothing$ is open: the condition holds automatically because we never have to test it (there is no x such that $x \in \varnothing$)! Intuitively, a subset of $\mathbf{R}$ is open if every point has "wiggle room": you can move a little bit and stay inside of the subset. We will call this the **Euclidean topology** on $\mathbf{R}$ and it is the default one for $\mathbf{R}$ if we don't specify. □

In this course, we'll focus more on "natural" examples and "good" properties, but it's good to keep in mind that the definition we've given is very flexible and general. For that reason it's not easy to have a good intuition about arbitrary topological spaces. For example, try to think about what "wiggle room" means here:

Example 1.2.6. Let $X = \{1, 2, 3\}$. Then $\mathcal{T} = \{\varnothing, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$ is a topology on X. □

Nonetheless, it is good practice to see how far we can go before specializing or tacking on extra properties. The notion of a topological space is fairly ubiquitous in diverse parts of mathematics and what is considered pathological in one application might be quite common in another.

Finally, it will be convenient to compare topologies on the same set.

Definition 1.2.7. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are both topologies on a set X, then we say that $\mathcal{T}_1$ **refines** $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$, that is, if being open in $\mathcal{T}_2$ implies being open in $\mathcal{T}_1$ (or another way, $\mathcal{T}_1$ has all of the open sets of $\mathcal{T}_2$ and possibly more). We will also say that $\mathcal{T}_2$ is **coarser** than $\mathcal{T}_1$. We will also write $\mathcal{T}_1 \geq \mathcal{T}_2$. □

Example 1.2.8. The discrete topology on X refines all topologies on X. The indiscrete topology is coarser than all topologies on X. So they are, respectively, the biggest and smallest topology you can put on a set in the language of the above notation. □

Remark 1.2.9. Keep in mind that given two topologies, they might not be comparable at all, i.e., it might be that neither refines the other (this is an example of a **partial ordering**).

For example, take $X = \{1, 2, 3\}$ and consider the topologies $\mathcal{T}_1 = \{\varnothing, \{1\}, \{1, 2\}, \{1, 2, 3\}\}$ and $\mathcal{T}_2 = \{\varnothing, \{3\}, \{2, 3\}, \{1, 2, 3\}\}$. Then neither refines the other. The topology $\mathcal{T}_3 = \{\varnothing, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$ refines $\mathcal{T}_1$ but is incomparable with $\mathcal{T}_2$.

We will often be interested in the coarsest topology on a set X satisfying some certain property (because refinement is a partial ordering, depending on the property, such a topology might not exist). □

1.3. Subbases and bases.

It is not always convenient to have to specify every single subset that is open. Instead, we might describe a smaller collection S of subsets and consider the topology that is "generated" by these subsets, i.e., the coarsest topology that contains S. For example, for $\mathbf{R}$, every open subset is a union of open intervals, so these open intervals "generate" the topology: the topology on $\mathbf{R}$ is the coarsest one that contains all of the open intervals. Let's make this idea more precise.

Definition 1.3.1. Let X be a set and let $S \subseteq 2^X$ be a collection of subsets.

Let $S' \subseteq 2^X$ be the collection of subsets which can be obtained by intersecting any finite number (including zero) of elements of S.

Finally, the **topology $\mathcal{T}$ generated by** S is denoted $\mathcal{T}(S)$, and is the collection of subsets which are unions (finite or not) of elements of S', and S is called a **subbasis** of $\mathcal{T}(S)$.

Remember that the union of 0 sets is empty and that the intersection of 0 sets is all of X, so $\mathcal{T}(S)$ contains both the empty set and X.[1] □

That is a mouthful, but to say it another way: first we take all possible intersections of finitely many elements of S, then we take all unions of the resulting subsets. What does an element of $\mathcal{T}(S)$ look like? It's a union, over some index set I, of finite intersections of elements of S. For $i \in I$, let n_i be the number of sets that are being intersected. So we can write it as

$$\bigcup_{i \in I} (\alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i})$$

where $\alpha_{i,k} \in S$.

We called the result a topology, but we should first prove that it has the required properties:

Proposition 1.3.2. *$\mathcal{T}(S)$ is a topology on X.*

Proof. We just need to check that the 4 axioms of a topology are satisfied:

(1) The empty set is in $\mathcal{T}(S)$ by the reminder given in the definition.
(2) Similarly, the whole set X is in $\mathcal{T}(S)$ because of the reminder in the definition.

[1] Munkres adds the requirement that the union of all of the elements of S gives X, because we want $X \in \mathcal{T}(S)$. But this is unnecessary because of the convention about empty intersections.

(3) We need to check that finite intersections of elements in $\mathcal{T}(S)$ are also in $\mathcal{T}(S)$. We actually just need to check that the intersection of two elements U_1 and U_2 in $\mathcal{T}(S)$ is also in $\mathcal{T}(S)$. From what we said above, we can write

$$U_1 = \bigcup_{i \in I} (\alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i})$$

where $\alpha_{i,k} \in S$ and

$$U_2 = \bigcup_{j \in J} (\beta_{j,1} \cap \cdots \cap \beta_{j,m_j})$$

where $\beta_{j,k} \in S$. We have the following formula for their intersection:

$$\bigcup_{i \in I} (\alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i}) \cap \bigcup_{j \in J} (\beta_{j,1} \cap \cdots \cap \beta_{j,m_j}) = \bigcup_{i \in I} \left(\bigcup_{j \in J} \alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i} \cap \beta_{j,1} \cap \cdots \cap \beta_{j,m_j} \right).$$

Finally, each term $\alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i} \cap \beta_{j,1} \cap \cdots \cap \beta_{j,m_j}$ is also an element of S'. The double union can be rewritten as a single union if we use $I \times J$ as our indexing set:

$$\bigcup_{(i,j) \in I \times J} (\alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i} \cap \beta_{j,1} \cap \cdots \cap \beta_{j,m_j}).$$

Hence the result is also an element in $\mathcal{T}(S)$.

(4) The idea for this one is actually easier than (3), but the notation is more complicated, so let's just say it in words. We're supposed to take a union of elements in $\mathcal{T}(S)$ and show it is also an element in $\mathcal{T}(S)$. But each element of $\mathcal{T}(S)$ is a union of elements in S'. But what is a union of unions? It's just an even larger union. (exercise: Write out this argument using notation like in (3). It might be confusing at first, but it's really useful to do carefully.) $\square$

In fact, $\mathcal{T}(S)$ is the coarsest topology that contains S (exercise: prove this). Said differently, any topology that has all of the elements of S as being open must be a refinement of $\mathcal{T}(S)$. So if we want everything in S to be an open set, we must make all of the elements of $\mathcal{T}(S)$ open as well, and this is the bare minimum that is needed.

Hence, every collection of subsets can be used to create a topology, but the resulting open subsets are obtained via a 2-step process.

Example 1.3.3. Go back to the case $X = \mathbf{R}$. We take our subbasis S to be the set of open intervals together with $\varnothing$. Then $\mathcal{T}(S)$ is the Euclidean topology. However, the open sets in $\mathbf{R}$ are unions of open intervals, there was no mention of taking intersections. That's because this example satisfies an extra property: the intersection of two open intervals is again an open interval, so we have $S = S'$. $\square$

Example 1.3.4. What topology do we get if we take $S = \varnothing$? What about if S only contains the singleton subset $\{x\}$ for some $x \in X$? More generally what if $S = \{\{x_1\}, \dots, \{x_n\}\}$ for some $x_1, \dots, x_n \in X$? $\square$

In general, we can avoid having to take intersections if our subbasis satisfies an extra condition (which is not as restrictive as the example above):

Definition 1.3.5. Let X be a set and $B \subseteq 2^X$. We call B a **basis** (for a topology on X), and the elements of B **basis elements** if the following condition holds:

(∗) For all $x \in X$, if x belongs to two basis elements b_1 and b_2, then there is a basis element b' such that $x \in b'$ and $b' \subseteq b_1 \cap b_2$. $\square$

Alternatively, condition $(*)$ just says that the intersection of any two basis elements can be written as the union of other basis elements.

By induction, condition $(*)$ is equivalent to the following statement (exercise: prove it):

($*'$) For all $x \in X$, if x belongs to n basis elements $b_1, \dots, b_n$, then there is a basis element b' such that $x \in b'$ and $b' \subseteq b_1 \cap \cdots \cap b_n$.

Condition $(*)$ allows us to avoid having to first construct the intermediate set S' of finite intersections as shown by the next result:

Proposition 1.3.6. *Let B be a basis. Every open set (except possibly X)*[2] *in the topology $\mathcal{T}(B)$ is a union of elements of B.*

Proof. As in the last proof, a general element $U \in \mathcal{T}(B)$ is of the form

$$U = \bigcup_{i \in I} (\alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i})$$

for some index set I and $\alpha_{i,k} \in B$. We want to argue that we can rewrite this in a way that does not use intersections. Focus on an element $i \in I$. Set $A_i = \alpha_{i,1} \cap \cdots \cap \alpha_{i,n_i}$. For each $x \in A_i$, using condition ($*'$), there exists a basis element, call it $\beta_{i,x}$, such that $x \in \beta_{i,x}$ and $\beta_{i,x} \subseteq A_i$. In other words, we have

$$A_i = \bigcup_{x \in A_i} \beta_{i,x}.$$

Hence we can rewrite U as

$$U = \bigcup_{i \in I} \bigcup_{x \in A_i} \beta_{i,x},$$

which can be written as a single union if we wanted, but the point is that it is nothing more than a union of basis elements. □

Warning 1.3.7. In linear algebra, a spanning set is a collection of vectors that allows us to obtain all vectors by taking linear combinations. In that sense, a basis for a topology is like a spanning set where "linear combination" is replaced by "taking unions". However it is **not** like a linear algebra basis which required everything to be obtained uniquely (different unions of basis elements in the topology sense could be equal, and this is allowed). □

Hence we can think of subbases and bases as compressing the information required for specifying a topology. We don't need to spell out what all of the open sets are, just certain ones, and the rest can be deduced via the operations of intersection and union (depending on which one we're dealing with). Bases are useful because we don't need to take intersections to get everything, but as for giving information, subbases can be much more compressed, so there is a tradeoff.

Example 1.3.8. Now we consider $X = \mathbf{R}^n$. For $x \in \mathbf{R}^n$ and a positive integer ϵ, the open ball centered at x of radius ϵ is

$$B(x, \epsilon) = \{y \in \mathbf{R}^n \mid |y - x| < \epsilon\},$$

i.e., all points at distance less than ϵ from x. This forms a basis (we'll check this in a more general context soon, but you might want to do it yourself), and the resulting topology is

[2] This is only because in our definition of subbasis, we do not assume that the union of all elements is X. This isn't really that important, but to be completely correct, we should point this out.

called the **Euclidean topology** on $\mathbf{R}^n$ and is the familiar one. When $n = 1$, the intersection of two open balls is again an open ball (they're just intervals), but this fails for $n > 1$. This is one example where it's useful not to require that the intersection of basis elements is again a basis element. □

1.4. **Subspaces.** Let X be a topological space and let $A \subseteq X$ be a subset.

Definition 1.4.1. Let $(X, \mathcal{T})$ be a topological space and $A \subseteq X$ a subset. The **subspace topology** $\mathcal{T}_A$ on A is defined by

$$\mathcal{T}_A = \{A \cap U \mid U \in \mathcal{T}\}.$$ □

As usual, we have to get this out of the way:

Proposition 1.4.2. *The subspace topology on A is a topology.*

Proof. We check the 4 conditions:

(1) $\varnothing \in \mathcal{T}_A$ since $\varnothing = A \cap \varnothing$.

(2) $A \in \mathcal{T}_A$ since $A = A \cap X$.

(3) Finite intersections: pick n elements of $\mathcal{T}_A$ which we write as $A \cap U_1, \dots, A \cap U_n$ where $U_i \in \mathcal{T}$. Then

$$(A \cap U_1) \cap \cdots \cap (A \cap U_n) = A \cap (U_1 \cap \cdots \cap U_n).$$

(4) Unions: pick an index set I and a collection of elements $\{A \cap U_i\}_{i \in I}$ from $\mathcal{T}_A$ where $U_i \in \mathcal{T}$ for all $i \in I$. Then

$$\bigcup_{i \in I}(A \cap U_i) = A \cap \bigcup_{i \in I} U_i.$$ □

Proposition 1.4.3. *Let B be a basis for the topology $\mathcal{T}$ on X. Then*

$$B_A = \{A \cap b \mid b \in B\}$$

is a basis for the subspace topology on A. Furthermore, $\mathcal{T}_A = \mathcal{T}(B_A)$.

Proof. To check that B_A is a basis, we need to check the condition $(*)$ of the definition. Suppose $x \in A$ and that $x \in A \cap b_1$ and $x \in A \cap b_2$ for some $b_1, b_2 \in B$. In particular, we also have $x \in b_1$ and $x \in b_2$, so since B is a basis for $\mathcal{T}$, there exists $b' \in B$ such that $x \in b'$ and $b' \subseteq b_1 \cap b_2$. But now we're done because $x \in A \cap b'$ and $A \cap b' \subseteq (A \cap b_1) \cap (A \cap b_2)$.

The proof that $\mathcal{T}_A = \mathcal{T}(B_A)$ is left as an exercise. □

This gives us an easy way to construct new topological spaces from ones that we already know. Our main example right now is $\mathbf{R}^n$ and taking subsets of $\mathbf{R}^n$ will be a good source of examples for us.

Example 1.4.4. For all $n \geq 0$, we define the n-**sphere** S^n to be the set of unit vectors in $\mathbf{R}^{n+1}$:

$$\mathrm{S}^n = \{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid \sqrt{x_1^2 + \cdots + x_{n+1}^2} = 1\}.$$

(The square root is unnecessary in this case.) This is given the subspace topology from $\mathbf{R}^{n+1}$.

- When $n = 0$, S^0 is two points: $\mathrm{S}^0 = \{1, -1\} \subseteq \mathbf{R}$. The subspace topology on S^0 is the just the discrete topology.
- When $n = 1$, S^1 is the familiar unit circle in $\mathbf{R}^2$.
- When $n = 2$, S^2 is the unit sphere in $\mathbf{R}^3$ and is what we usually think of as a sphere.

We number them this way because S^n is "n-dimensional". We won't discuss what that means precisely (this topic is covered in Math 150B), but you can intuitively understand that from the examples above. $\square$

Example 1.4.5. For all $n \geq 1$, the n-**dimensional ball** B^n is the set of vectors with length at most 1 in $\mathbf{R}^n$:

$$\mathrm{B}^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid \sqrt{x_1^2 + \cdots + x_n^2} \leq 1\}.$$

(The square root is unnecessary in this case.) This is given the subspace topology from $\mathbf{R}^n$ and is essentially the "inside" of S^{n-1}:

- When $n = 1$, B^1 is the interval $[-1, 1]$.
- When $n = 2$, B^1 is the unit disk in $\mathbf{R}^2$.

Again, B^n is "n-dimensional", which is why we index them in this way. $\square$

Example 1.4.6. We can approximately think of the world around us as being $\mathbf{R}^3$ so it has the Euclidean topology. Hence every object around us (like you or your desk, ...) inherits the subspace topology from $\mathbf{R}^3$. $\square$

Remark 1.4.7. If we have a topological space, we might wonder if it can be realized as a subspace of $\mathbf{R}^n$ for some n so that its topology is just the subspace topology from $\mathbf{R}^n$ (this is an instance of an "embedding problem"). In general, the answer is no because being a subspace of $\mathbf{R}^n$ forces extra properties on the topology, but time permitting, we'll discuss when the answer is yes. $\square$

Finally, we need to address a possible ambiguity. If $A \subseteq X$ is a subset, then subsets of A are also subsets of X. However, a set which is open in A might not be open in X:

Example 1.4.8. Take $X = \mathbf{R}$ and $A = [-1, 1]$. Then the set $U = (0, 1]$ is open in A because $(0, 1] = A \cap (0, 2)$, for example. But it's not open in $\mathbf{R}$: there is no open interval centered at 1 which stays inside of U. $\square$

So we should introduce new terminology. If $U \in \mathcal{T}_A$, then we will say that U is **open in** A, or **open relative to** A. As we just saw, being open in A does not imply being open in X. There is a special case where we don't need to separate these notions:

Proposition 1.4.9. *If $A \subseteq X$ is an open set, then for $U \subseteq A$, U is open in A if and only if U is open in X.*

Proof. If U is open in X, then we also know that U is open in A because we can write $U = A \cap U$. Conversely, if U is open in A, then by definition, we have $U = A \cap V$ for some open set V in X. But A is also open in X, so the same is true for U since it is the intersection of two open sets. $\square$

1.5. Closed sets.

So far, to "know" a topological space, we need to know which subsets are considered open. The obvious way to do that is to have a list of all of them. But you could also explain that information in a different way. Here's an example.

Definition 1.5.1. If X is a topological space, then a subset $Z \subseteq X$ is called **closed** if its complement $Z^c = X \setminus Z$ is open. $\square$

Example 1.5.2. For $\mathbf{R}$, the "closed intervals" $[a, b] = \{x \mid a \leq x \leq b\}$ are closed: its complement is $(-\infty, a) \cup (b, \infty)$ which is open. $\square$

So if you know what all of the closed subsets are, then you can also determine what the open subsets are (just take their complements).

By translating the 4 axioms for open sets, we get some basic properties for closed sets:

Proposition 1.5.3. *Let X be a topological space. Then:*

(1) $\varnothing$ is closed.
(2) X is closed.
(3) The union of finitely many closed sets is closed.
(4) The intersection of any collection of closed sets is closed.

Proof. (1) $\varnothing = X \setminus X$.

(2) $X = X \setminus \varnothing$.

(3) If $Z_1, \ldots, Z_n$ are closed, then $U_i = X \setminus Z_i$ is open and

$$Z_1 \cup \cdots \cup Z_n = (X \setminus U_1) \cup \cdots \cup (X \setminus U_n) = X \setminus (U_1 \cap \cdots \cap U_n)$$

and $U_1 \cap \cdots \cap U_n$ is open.

(4) If I is an index set and $\{Z_i\}_{i \in I}$ is a collection of closed sets, set $U_i = X \setminus Z_i$. We have

$$\bigcap_{i \in I} Z_i = \bigcap_{i \in I} (X \setminus U_i) = X \setminus (\bigcup_{i \in I} U_i)$$

and $\bigcup_{i \in I} U_i$ is open. □

Remark 1.5.4. In fact, we could have replaced Definition 1.2.1 (let's call this the "open set definition") with a definition that told us which subsets are closed (let's call this the "closed set definition") and use the properties from the previous result as our axioms.

Is there any reason to prefer the "open set definition" over the "closed set definition"? Not really, it's just a matter of convention. Some things might be easier to define in terms of open sets while others might be easier in terms of closed sets. We'll just do whatever is most convenient. □

Warning 1.5.5. In everyday language, "open" is the opposite of "closed", i.e., something can't be both open and closed, but this is not true in topology! For instance, the empty set is always open, and it's always closed (same with X). In the discrete topology, every subset is open and also closed! It's a rather unfortunate use of language, but it's so ingrained that we just have to deal with it. A set which is both open and closed is called **clopen**. □

Let's see how closed sets interact with subspaces. If $A \subseteq X$ has the subspace topology, then by analogy with open sets, we'll say that a subset Z of A is **closed in** A, or **closed relative to** A, if it is a closed set in the subspace topology on A.

Proposition 1.5.6. *Let $A \subseteq X$ be equipped with the subspace topology. A subset Z of A is closed in A if and only if there exists a closed subset Z' of X such that $Z = A \cap Z'$.*

Proof. First suppose that Z is closed in A and set $U = A \setminus Z$. Then U is open in A, so there exists an open set U' of X such that $U = A \cap U'$. But

$$Z = A \setminus U = A \setminus (A \cap U') = A \cap (X \setminus U')$$

and $X \setminus U'$ is closed in X. Conversely, suppose that there exists a closed subset Z' of X such that $Z = A \cap Z'$. But then $A \setminus Z = A \cap (X \setminus Z')$ and $X \setminus Z'$ is open in X, so $A \setminus Z$ is open in A, which means that Z is closed in A. □

Proposition 1.5.7. *If A is closed in X, then for $Z \subseteq A$, Z is closed in A if and only if Z is closed in X.*

Proof. Same as Proposition 1.4.9. □

1.6. Closure and interior.

Definition 1.6.1. Let X be a topological space and let A be a subset of X.

The **interior** of A is denoted A°, or $\mathrm{Int}_X(A)$, and is the union of all open subsets of X that are also subsets of A.

The **closure** of A is denoted $\overline{A}$, or $\mathrm{Cl}_X(A)$, and is the intersection of all closed subsets of X that contain A. □

Just some basic properties that follow from the definitions:

- A° is open and $A^\circ \subseteq A$. Furthermore, it contains every other open subset that is contained in A. Another way to say this is: A° is the largest open subset inside of A.
- $\overline{A}$ is closed and $\overline{A} \supseteq A$. Furthermore, it is contained in every other closed subset that contains A. Another way to say this is: $\overline{A}$ is the smallest closed subset that contains A.
- If $B \subseteq A$, then $\overline{B} \subseteq \overline{A}$ and $B^\circ \subseteq A^\circ$.

Warning 1.6.2. To make sure we understand "largest open set" and "smallest closed set", consider a variation. We might want to consider the "largest closed set" contained in A or the "smallest open set" containing A instead. But actually it might not exist. For example, take $X = \mathbf{R}$ and $A = [0, 1]$. For every open set that contains A, you can always find a smaller one that also contains A (why?). Similarly, if $A' = (0, 1)$, then for every closed set in A', you can always find a larger one that is also in A' (why?). □

It will follow from a homework problem that

$$A^\circ = X \setminus \overline{(X \setminus A)}, \qquad \overline{A} = X \setminus (X \setminus A)^\circ$$

so if we study one of these operations, we technically can translate them to the other one. We'll pick closure to focus on.

Let's give another way to characterize closures. First, some definitions:

Definition 1.6.3. If $x \in X$ is a point and U is an open subset that contains x, we say that U is a **neighborhood** of x. □

Definition 1.6.4. If two subsets A and B have nonempty intersection, then we say that A **intersects** B. □

Proposition 1.6.5. *Let A be a subset of a topological space X. Then $x \in \overline{A}$ if and only if every neighborhood of x intersects A.*

Proof. First assume that $x \in \overline{A}$. Let U be a neighborhood of x. Suppose, for the sake of contradiction, that U does not intersect A. But then $X \setminus U$ is a closed set that contains A. In particular, $\overline{A} \subseteq X \setminus U$. But $x \in \overline{A}$ and $x \notin X \setminus U$, so we have a contradiction. We have just shown that every neighborhood of x intersects A.

Conversely, now assume that every neighborhood of x intersects A. To show that $x \in \overline{A}$, we need to argue that every closed set that contains A must also contain x. So let Z be a closed set that contains A. Again, for contradiction's sake, suppose that $x \notin Z$. But then $X \setminus Z$ is a neighborhood of x, and our assumption implies that $X \setminus Z$ intersects A. But that contradicts the fact that Z contains A. We conclude that $x \in \overline{A}$. □

Corollary 1.6.6. *Let Y be a subspace of X and let A be a subset of Y. Then* $\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$.

To be clear, $\mathrm{Cl}_X(A)$ is the closure of A when thought of as a subset in the topological space X, and $\mathrm{Cl}_Y(A)$ is the closure of A when thought of as a subset of Y, given the subspace topology from X. These are different things, but the result tells us that they are very closely related.

Proof. Pick $x \in \mathrm{Cl}_Y(A)$. By definition, $x \in Y$. Let U be a neighborhood of x in X. By definition, $Y \cap U$ is a neighborhood of x in the subspace topology for Y, and it intersects A by Proposition 1.6.5. Hence U also intersects A, and since U was general, we again use Proposition 1.6.5 to conclude that $x \in \mathrm{Cl}_X(A)$.

Conversely, suppose that $x \in \mathrm{Cl}_X(A) \cap Y$. A general neighborhood of x in Y is of the form $Y \cap U$ where U is a neighborhood of x in X. By Proposition 1.6.5, U intersects A. But since $A \subseteq Y$, we can also say that $Y \cap U$ intersects A since $(Y \cap U) \cap A = U \cap A$. Using Proposition 1.6.5 one more time, we conclude that $x \in \mathrm{Cl}_Y(A)$. $\square$

Example 1.6.7. Take $X = \mathbf{R}$ and $Y = (-1, 1)$ and $A = (0, 1)$. Then $\mathrm{Cl}_X(A) = [0, 1]$ and $\mathrm{Cl}_Y(A) = [0, 1)$. $\square$

Definition 1.6.8. Let A be a subset of a topological space X. A point x is a **limit point** (also less commonly called **cluster point** or **point of accumulation**) of A if $x \in \overline{A \setminus \{x\}}$. Note that we do not require that $x \in A$ (if $x \notin A$ then $A \setminus \{x\} = A$). $\square$

Using Proposition 1.6.5, x is a limit point if and only if every neighborhood of x intersects $A \setminus \{x\}$.

Proposition 1.6.9. *Let A' be the set of limit points of A. Then*

$$\overline{A} = A \cup A'.$$

Proof. We will first show that $A \cup A' \subseteq \overline{A}$. By definition, we have $A \subseteq \overline{A}$, so we just need to show that $A' \subseteq \overline{A}$. So pick a point $x \in A'$. By definition, $x \in \overline{A \setminus \{x\}}$, but $\overline{A \setminus \{x\}} \subseteq \overline{A}$, so $x \in \overline{A}$, and we have shown that $A' \subseteq \overline{A}$.

Now we show the reverse inclusion $\overline{A} \subseteq A \cup A'$. For that, pick $x \in \overline{A}$. If $x \in A$, then we're done, so let's focus on the case that $x \notin A$. But then $A \setminus \{x\} = A$, and so $x \in \overline{A} = \overline{A \setminus \{x\}}$, which means $x \in A'$ by definition. $\square$

This lets us connect the ideas of closed sets and limit points:

Corollary 1.6.10. *A is closed if and only if it contains all of its limit points.*

Proof. By definition, A is closed if and only if $A = \overline{A}$. By the previous formula, the condition $A = \overline{A}$ is equivalent to $A' \subseteq A$, i.e., A contains all of its limit points. $\square$

1.7. **Continuous functions.** It will be important to consider functions between topological spaces, and especially those that preserve the topology. Continuous functions are the standard way to do that. By analogy with abstract algebra, these play the role of homomorphisms.

Given a function $f \colon A \to B$ and a subset $S \subseteq B$, recall that the **preimage** of S is

$$f^{-1}(S) = \{a \in A \mid f(a) \in S\}.$$

Definition 1.7.1. Let X and Y be topological spaces. A function $f\colon X \to Y$ is **continuous** if, for every open subset $U \subseteq Y$, the preimage $f^{-1}(U) \subseteq X$ is also open. $\square$

At this point, it's natural to ask if this coincides with other definitions of continuous, like in calculus (when $X = Y = \mathbf{R}$). It does, but it will be easier to explain this after developing a few more definitions and results.

Also, we never have to check the condition in this definition for $U = \varnothing$ and $U = Y$ because $f^{-1}(\varnothing) = \varnothing$ and $f^{-1}(Y) = X$. This doesn't usually save us any time, but it's good to point out.

Remark 1.7.2. It's probably not clear why this is a good definition (wouldn't it be more natural to ask that if $V \subseteq X$ is open then so is $f(V) \subseteq Y$?). I don't have a good explanation for why this is what we want, but it turns out to work out really nicely, so we'll accept for the moment that this is the better idea to focus on. $\square$

Note that if X has the discrete topology, then *any* function $f\colon X \to Y$ for any topological space Y is continuous. Similarly, if Y has the indiscrete topology, then *any* function $f\colon X \to Y$ for any topological space X is continuous.

Remark 1.7.3. The intuition here is that the more open sets X has, the easier it is for functions out of it to be continuous, and the less open sets Y has, the easier it is for functions into it to be continuous. $\square$

Proposition 1.7.4. *The composition of continuous functions is continuous, i.e., if $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then so is $g \circ f\colon X \to Z$.*

Proof. Let $U \subseteq Z$ be an open subset. Since g is continuous, $g^{-1}(U) \subseteq Y$ is open. Since f is continuous, $f^{-1}(g^{-1}(U)) \subseteq X$ is also open. Finally,

$$\begin{aligned} f^{-1}(g^{-1}(U)) &= \{x \in X \mid f(x) \in g^{-1}(U)\} \\ &= \{x \in X \mid g(f(x)) \in U\} \\ &= (g \circ f)^{-1}(U). \end{aligned}$$

So we have shown that for every open subset of Z, its preimage under $g \circ f$ is also open, which means that $g \circ f$ is continuous. $\square$

Proposition 1.7.5. *Let B be a basis for the topology on Y and let $f\colon X \to Y$ be a function. Then f is continuous if and only if, for all $b \in B$, we have that $f^{-1}(b)$ is open.*

Proof. First suppose that f is continuous. Since each $b \in B$ is in fact open, we must have that $f^{-1}(b)$ is open.

Now suppose that for all $b \in B$, we have that $f^{-1}(b)$ is open. Let $U \subseteq Y$ be an open subset not equal to Y. Then we can write U as a union of basis elements, write it as $U = \bigcup_{i \in I} b_i$ for some index set I. But then

$$f^{-1}(U) = \bigcup_{i \in I} f^{-1}(b_i).$$

By our assumption, each $f^{-1}(b_i)$ is open, and the union of open sets is again open. In particular, $f^{-1}(U)$ is open. Since U was arbitrary, this shows that f is continuous. $\square$

It will be convenient to have different ways to check that a function is continuous. First, we need a definition.

Definition 1.7.6. Let $f\colon X \to Y$ be a function between topological spaces and pick $x \in X$. We say that f is **continuous at** x if, for each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$. □

Example 1.7.7. What does this say when $X = Y = \mathbf{R}$? I claim that is equivalent to the usual δ-ϵ definition of continuity at a point x. For now we'll call it "RA (real analysis) continuous" to distinguish it from our current definition.

Recall what that says: a function $f\colon \mathbf{R} \to \mathbf{R}$ is RA continuous at x, if for all $\epsilon > 0$, there exists $\delta > 0$ such that $f((x-\delta, x+\delta)) \subseteq (f(x)-\epsilon, f(x)+\epsilon)$ (I've used interval notation to make the translation smoother).

If we assume that a function is continuous at x, then it is RA continuous at x: $(f(x) - \epsilon, f(x)+\epsilon)$ is a neighborhood of $f(x)$ and $(x-\delta, x+\delta)$ is a neighborhood of x.

What about the converse? If f is RA continuous at x, is it continuous at x? Yes: pick any neighborhood V of $f(x)$. By definition, V must contain an open interval that contains $f(x)$, and if we shrink it appropriately, we may assume that it is centered at $f(x)$, so it is of the form $(f(x)-\epsilon, f(x)+\epsilon)$ for some $\epsilon > 0$. Now since f is RA continuous at x, there exists $\delta > 0$ such that $f((x-\delta, x+\delta)) \subseteq (f(x)-\epsilon, f(x)+\epsilon) \subseteq V$. Since V was arbitrary, f is continuous at x. □

What we'd like to know is: if f is continuous at x for all $x \in X$, is that the same as f being continuous? We'll prove that in the next result, and when specialized to $X = Y = \mathbf{R}$, it says that our general topology definition of continuous agrees with the calculus definition (so we can stop saying "RA continuous").

Theorem 1.7.8. *Let $f\colon X \to Y$ be a function between topological spaces X and Y. The following statements are equivalent:*

(1) f is continuous.
(2) For every subset $A \subseteq X$, we have that $f(\overline{A}) \subseteq \overline{f(A)}$.
(3) For every closed subset $Z \subseteq Y$, we have that $f^{-1}(Z)$ is closed in X.
(4) For every $x \in X$, we have that f is continuous at x.

Proof. (1) $\Longrightarrow$ (2): First assume that f is continuous. We will show that this implies (2). So pick a subset $A \subseteq X$ and pick $x \in \overline{A}$. We need to show that $f(x) \in \overline{f(A)}$, and by Proposition 1.6.5, it is enough to show that every neighborhood of $f(x)$ intersects $f(A)$. Let U be a neighborhood of $f(x)$. Since f is continuous, $f^{-1}(U)$ is open, and hence a neighborhood of x, and so it intersects A (again by Proposition 1.6.5). So we can find an element $y \in f^{-1}(U) \cap A$. But then $f(y) \in U \cap f(A)$, so we see that U intersects $f(A)$.

(2) $\Longrightarrow$ (3): Now suppose that (2) holds; we will show it implies (3). Let $Z \subseteq Y$ be a closed subset and define $A = f^{-1}(Z)$, so that $f(A) \subseteq Z$. Using (2), we know that $f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{Z}$, but since Z is closed, we have $\overline{Z} = Z$, so in fact $f(\overline{A}) \subseteq Z$.

Now we claim that $A = \overline{A}$. If $x \in \overline{A}$, then $f(x) \in Z$, so by definition, we have $x \in f^{-1}(Z) = A$. This proves the claim, and so the inverse image of Z is closed.

(3) $\Longrightarrow$ (4): Now we show that (3) implies (4). Pick a point $x \in X$ and a neighborhood V of $f(x)$. Define $Z = Y \setminus V$, which is closed. By (3), $f^{-1}(Z)$ is also closed. Now set $U = X \setminus f^{-1}(Z)$, which is open. Note that U contains x: if not, then $f(x) \in Z$, but that's not the case. To finish showing that f is continuous at x, it suffices to show that $f(U) \subseteq V$. If $y \in U$, then $f(y) \notin Z$, which means that $f(y) \in V$, which is what we want.

(4) $\implies$ (1): Finally, we show that (4) implies (1). So pick an open set $U \subseteq Y$. We want to show that $f^{-1}(U)$ is open. If $f^{-1}(U) = \varnothing$, we can stop, otherwise, pick $x \in f^{-1}(U)$. Then U is a neighborhood of $f(x)$. Since f is continuous at x by (4), there is a neighborhood of x, let's call it α_x, such that $f(\alpha_x) \subseteq U$. This also means that $\alpha_x \subseteq f^{-1}(U)$. This discussion implies that

$$f^{-1}(U) = \bigcup_{x \in f^{-1}(U)} \alpha_x$$

and now we're done because a union of open sets is again open. □

Now we discuss how continuous functions interact with subspaces.

Proposition 1.7.9. *Let A be a subspace of X.*

(1) *The inclusion $i\colon A \to X$ defined by $i(a) = a$ is continuous.*
(2) *(Restricting codomain) If $f\colon Z \to X$ is any continuous function whose image is a subset of A, then the function $g\colon Z \to A$ defined by $g(z) = f(z)$ is also continuous.*
(3) *(Expanding codomain) Conversely, if $g\colon Z \to A$ is a continuous function, then the function $f\colon Z \to X$ given by $f(z) = g(z)$ is also continuous.*
(4) *(Restricting domain) If $f\colon Z \to X$ is continuous and $B \subseteq Z$ is a subspace, then the restriction $f|_B\colon B \to X$ defined by $f|_B(x) = f(x)$ for all $x \in B$ is also continuous.*

Proof. (1) Let U be any open set in X. Then $i^{-1}(U) = A \cap U$ is, by definition, open in A. Hence i is continuous.

(2) Any open set in A can be written as $A \cap U$ where U is open in X. Then $g^{-1}(A \cap U) = f^{-1}(U)$ because if $x \in U \setminus A$, then $f^{-1}(x)$ is empty by assumption. But $f^{-1}(U)$ is open since f is continuous, so we see that g is also continuous.

(3) We have $f = i \circ g$ where $i\colon A \to X$ is the inclusion map, so f is continuous by (1) together with Proposition 1.7.4.

(4) Let $i\colon B \to Z$ be the inclusion, which we have seen is continuous. Then $f|_B = f \circ i$. □

Example 1.7.10. Consider the function $f\colon \mathbf{R} \to \mathbf{R}^2$ given by $f(x) = (\cos x, \sin x)$. From basic trigonometry, we know that the image of f is contained in the unit circle S^1. You know from calculus that f is continuous (we'll discuss a bit more about this later). Hence the previous result says that the associated function $g\colon \mathbf{R} \to \mathrm{S}^1$ with $g(x) = (\cos x, \sin x)$ is also continuous. □

Definition 1.7.11. Let I be an index set and $\{U_i\}_{i \in I}$ a collection of open sets in X. Then $\{U_i\}$ is an **open covering of** X if $\bigcup_{i \in I} U_i = X$. □

Proposition 1.7.12 (Local formation of continuity)**.** *Let $\{U_i\}_{i \in I}$ be an open covering of X and equip each U_i with the subspace topology. Let Y be a topological space and let $f\colon X \to Y$ be a function. Then f is continuous if and only if the restriction $f|_{U_i}\colon U_i \to Y$ is continuous for all $i \in I$.*

Proof. First assume that f is continuous. Then for all $i \in I$, $f|_{U_i}$ is continuous by Proposition 1.7.9(4).

Conversely, assume that each $f|_{U_i}$ is continuous. Let $V \subset Y$ be an open set. Then

$$f^{-1}(V) = \bigcup_{i \in I} f^{-1}(V) \cap U_i$$

because $\bigcup_{i\in I} U_i = X$. Next $f^{-1}(V) \cap U_i = (f|_{U_i})^{-1}(V)$ and hence is open in U_i. Since U_i is open in X, there is no difference between being open in U_i or being open in X (see Proposition 1.4.9) and so we conclude that $f^{-1}(V) \cap U_i$ is open in X for all $i \in I$. But then $f^{-1}(V)$ is a union of open sets, so is itself open. $\square$

Finally, here's a useful way to define continuous functions in a piecewise way.

Lemma 1.7.13 (Pasting lemma)**.** *Suppose $A_1, A_2 \subseteq X$ are closed subsets such that $A_1 \cup A_2 = X$. Let Y be a topological space and let $f_1 \colon A_1 \to Y$ and $f_2 \colon A_2 \to Y$ be continuous functions such that for all $x \in A_1 \cap A_2$, we have $f_1(x) = f_2(x)$. Then define $g \colon X \to Y$ by $g(x) = f_1(x)$ if $x \in A_1$ and $g(x) = f_2(x)$ if $x \in A_2$ (by our assumption, there is no ambiguity if $x \in A_1 \cap A_2$). Then g is also continuous.*

Proof. By Theorem 1.7.8, it is enough to show that if $Z \subseteq Y$ is any closed subset, then $g^{-1}(Z)$ is also closed. So let Z be a closed subset. Then $g^{-1}(Z) = f_1^{-1}(Z) \cup f_2^{-1}(Z)$ because $A_1 \cup A_2 = X$. Since f_1, f_2 are continuous, $f_i^{-1}(Z)$ is closed in A_i for $i = 1, 2$, and hence also closed in X by Proposition 1.5.7. So $g^{-1}(Z)$ is also closed in X and we're done. $\square$

We can extend the previous result to n closed subsets $A_1, \dots, A_n$ and the same proof works, but we just did $n = 2$ to make the notation simpler.

Example 1.7.14. You've seen this when defining piecewise functions. For example, consider the function $g \colon \mathbf{R} \to \mathbf{R}$ defined by

$$g(x) = \begin{cases} x & \text{if } x \leq 0 \\ x^2 & \text{if } x \geq 0 \end{cases}.$$

This fits into the pasting lemma situation by taking $A_1 = (-\infty, 0]$ and $A_2 = [0, \infty)$ with $f_1(x) = x$ and $f_2(x) = x^2$. Then $A_1 \cap A_2 = \{0\}$ and $f_1(0) = 0 = f_2(0)$. Since both f_1 and f_2 are continuous, we conclude that g is also continuous. $\square$

1.8. **Homeomorphisms and embeddings.** Finally, we come to the notion of "isomorphism". But we call it something different in topology:

Definition 1.8.1. Let X, Y be topological spaces. A function $f \colon X \to Y$ is a **homeomorphism** if:

(1) f is continuous,
(2) f is a bijection, and
(3) its inverse f^{-1} is also continuous. $\square$

If a homeomorphism exists between X and Y, then we say that X and Y are **homeomorphic** and also write $X \cong Y$.

Warning 1.8.2. In abstract algebra, if f is a bijective homomorphism between two groups, then its inverse f^{-1} is automatically a homomorphism, so we might think that (3) above is superfluous. But this is not so, i.e., a continuous bijection could have an inverse which fails to be continuous. See the next example. $\square$

Example 1.8.3. Let X be any set with at least 2 elements and consider the identity function $f \colon X \to X$. If the first X is given the discrete topology and the second X is given any topology that is not the discrete topology (like the indiscrete topology), then f is automatically continuous but f^{-1} will not be. $\square$

Proposition 1.8.4. *Let $f\colon X \to Y$ be a continuous bijection. Then f is a homeomorphism if and only if, for all open subsets $U \subseteq X$, the image $f(U)$ is also open in Y.*

Proof. To avoid confusion, let g be the inverse of f. Then for any subset $U \subseteq X$, we have $f(U) = g^{-1}(U)$. In particular, f is a homeomorphism if and only if g is continuous (it is the only extra condition we need to check), and by the identity $f(U) = g^{-1}(U)$, this is equivalent to $f(U)$ being open for any open subset U. □

Proposition 1.8.5. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be homeomorphisms. Then:*
(a) *$f^{-1}\colon Y \to X$ is a homeomorphism.*
(b) *$g \circ f\colon X \to Z$ is a homeomorphism.*

Proof. (a): First we check that f^{-1} is also a homeomorphism, so we need to check that the 3 conditions in the definition are true:
(1) Since f is a homeomorphism, we are guaranteed that f^{-1} is continuous by definition.
(2) Since f is a bijection, the same is true for f^{-1}.
(3) Finally, $(f^{-1})^{-1} = f$, which is continuous because f is a homeomorphism.

(b): Now we check that $g \circ f$ is a homeomorphism, again by checking that the 3 conditions in the definition are true:
(1) $g \circ f$ is continuous because f, g both are (now use Proposition 1.7.4).
(2) The composition of bijective functions is again bijective.
(3) Finally, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, and both f^{-1} and g^{-1} are continuous since f, g are homeomorphisms, so again we use Proposition 1.7.4 to conclude that $f^{-1} \circ g^{-1}$ is continuous. □

Definition 1.8.6. Let $f\colon X \to Y$ be an injective (i.e., one-to-one) continuous function between topological spaces. We say that f is an **embedding** if it defines a homeomorphism of X with its image $f(X)$, where $f(X)$ is given the subspace topology from Y. □

If f is an embedding, then we are free to identify X and its image since they have the same topology.

Example 1.8.7. Consider the function $f\colon \mathbf{R} \to \mathbf{R}^2$ given by $f(x) = (x, x^2)$. This is continuous and injective and is also an embedding. For that, we just need to check that f sends open sets to open sets in the image of f (which is a parabola) and actually we only need to check that f sends the basis of open intervals to open sets. Let's consider an open interval (a, b). The image is the set $\{(x, x^2) \mid a < x < b\}$. This is the intersection of the parabola with the set $\{(x, y) \mid a < x < b\}$ which is open (check it). □

Example 1.8.8. Consider the function $f\colon [0, 2\pi) \to \mathbf{R}^2$ given by $f(x) = (\sin x, \cos x)$. This is continuous and injective, but is not an embedding. For example, the subset $[0, \pi/2)$ is open in $[0, 2\pi)$. (no picture here, but will draw in lecture) Its image under f contains the point $(1, 0)$ but no points in the southeast quadrant. However, for any $\epsilon > 0$, the ϵ-ball centered at $(1, 0)$ will always contain a point on the unit circle which is in the southeast quadrant (this is a little messy so we'll avoid doing the calculation rigorously, but you should check it). □

2. Basic constructions

We'll go through some ways to build new topological spaces using existing ones. Throughout, we want to keep in mind the intuition from Remark 1.7.3. While the definitions may seem arbitrary at first glance, they become more natural if we keep in mind that they are defined that way so that certain functions are automatically continuous. These are examples of "universal properties" which you may learn about in more detail in later courses.

2.1. **Product spaces.** Let I be an index set and suppose we are given a topological space X_i for each $i \in I$. We define the **product** $\prod_{i\in I} X_i$ to be the set of tuples $(x_i)_{i\in I}$ where $x_i \in X_i$. When $I = \{1,\dots,n\}$, this is the usual product $X_1 \times \cdots \times X_n$.

Our goal is to discuss how to put a topology on the product.

Given $j \in I$, the **projection map**

$$\pi_j \colon \prod_{i\in I} X_i \to X_j, \qquad \pi_j((x_i)_{i\in I}) = x_j$$

simply picks out the element that is indexed by j. Next, suppose we are given subsets $A_i \subseteq X_i$ for all $i \in I$. Then we can think of the product $\prod_{i\in I} A_i$ as a subset of $\prod_{i\in i} X_i$.

Definition 2.1.1. Notation as above, let S be the collection of subsets of $\prod_{i\in I} X_i$ which are of the form $\pi_j^{-1}(U_j)$ for some $j \in I$ and for some open subset $U_j \subseteq X_j$. The topology $\mathcal{T}(S)$ generated by S is called the **product topology** on $\prod_{i\in I} X_i$. $\square$

If we define $V_i = X_i$ for $i \neq j$ and $V_j = U_j$, then $\pi_j^{-1}(U_j) = \prod_{i\in I} V_i$.

Example 2.1.2. Suppose that $I = \{1,2\}$ so we're just dealing with two spaces X_1, X_2. Given an open subset $U_1 \subseteq X_1$, we have $\pi_1^{-1}(U_1) = U_1 \times X_2$ and similarly, for an open subset $U_2 \subseteq X_2$, we have $\pi_2^{-1}(U_2) = X_1 \times U_2$. Hence our set S is all subsets which are either of the form $U_1 \times X_2$ or $X_1 \times U_2$ where U_1, U_2 are open. $\square$

Remark 2.1.3. Is the Euclidean topology on $\mathbf{R}^n$ the same as the product topology (of n copies of $\mathbf{R}$)? It is, but we'll postpone that proof until the next section. $\square$

For the next result, given a collection of subsets $U_i \subseteq X_i$ for $i \in I$, we say that "$U_i = X_i$ for all but finitely many i" if the set $\{j \in I \mid U_j \neq X_j\}$ is finite.

Proposition 2.1.4. *Let B be the collection of subsets of $\prod_{i\in I} X_i$ of the form $\prod_{i\in I} U_i$ where $U_i \subseteq X_i$ is open, and $U_i = X_i$ for all but finitely many i. Then B is a basis for the product topology.*

Proof. Keep the notation from the definition, so that the product topology is $\mathcal{T}(S)$ where S is the set defined above. We need to check that B is a basis, and that $\mathcal{T}(B) = \mathcal{T}(S)$.

First, let's check that B is a basis using the definition. So pick two elements $\prod_{i\in I} U_i$ and $\prod_{i\in I} U_i'$ in B. Then

$$\left(\prod_{i\in I} U_i\right) \cap \left(\prod_{i\in I} U_i'\right) = \prod_{i\in I} (U_i \cap U_i').$$

Also

$$\{j \in I \mid U_j \cap U_j' \neq X_j\} = \{j \in I \mid U_j \neq X_j\} \cup \{j \in I \mid U_j' \neq X_j\}$$

is a union of two finite sets and hence is also finite. This shows that the intersection of two elements in B is again an element in B. This automatically implies that B is a basis (in the notation of the definition, we just take $b' = b_1 \cap b_2$).

Now we need to check that $\mathcal{T}(B) = \mathcal{T}(S)$. Actually we'll prove that $B = S'$, the set of finite intersections of elements in S, which implies this. Pick a finite subset J of I and open sets $U_j \subseteq X_j$ for $j \in J$. Then if we set $V_j = U_j$ for $j \in J$ and $V_k = X_k$ for $k \notin J$, then

$$\bigcap_{j \in J} \pi_j^{-1}(U_j) = \prod_{i \in I} V_i,$$

so $S' = B$. $\square$

Here's the main reason why this is a good definition.

Proposition 2.1.5. *(1) The projection maps $\pi_j \colon \prod_{i \in I} X_i \to X_j$ are continuous for all $j \in I$.*

(2) Let Y be a topological space and suppose we are given functions $f_i \colon Y \to X_i$ for all $i \in I$. Define $f \colon Y \to \prod_{i \in i} X_i$ by $f(y) = (f_i(y))_{i \in I}$. Then f is continuous if and only if, for all $i \in I$, f_i is continuous.

Proof. (1) Let $U \subseteq X_j$ be an open subset. Then $\pi_j^{-1}(U)$ is, by definition, open in $\prod_{i \in I} X_i$.

(2) First suppose that f is continuous. Then $f_i = \pi_i \circ f$, and hence it is a composition of continuous functions.

Conversely, suppose that f_i is continuous for all $i \in I$. We have defined a basis B in the previous result. To show that f is continuous, it suffices to show that for all $b \in B$, we have that $f^{-1}(b)$ is open in Y. We can write $b = \prod_{i \in I} U_i$ where $U_i \subseteq X_i$ is open and $U_i = X_i$ for all but finitely many i. Then $f^{-1}(b) = \bigcap_{i \in I} f_i^{-1}(U_i)$. If $U_i = X_i$, then $f_i^{-1}(U_i) = Y$, and so we can exclude that term from the intersection without affecting it. So in fact, $f^{-1}(b)$ is an intersection of *finitely many* sets of the form $f_i^{-1}(U_i)$. Each one is open since f_i is continuous, so we see that $f^{-1}(b)$ is also open. $\square$

Remark 2.1.6. The above result is about functions being continuous where the product is the domain (first part) and where it is the codomain (second part). Our general heuristic is that having more open sets makes it easier for the first part to be true, but that would make it harder for the second part to be true. So actually, we see that the product topology is "optimal" and has exactly enough open sets for both statements to be true. $\square$

Example 2.1.7. In Proposition 2.1.4, we could instead consider the collection of subsets of the form $\prod_{i \in I} U_i$ where $U_i \subseteq X_i$ is open, with no further conditions. This also forms a basis, and generates a topology on $\prod_{i \in I} X_i$ called the **box topology**. If I is finite, then there is no difference, but otherwise, it refines the product topology and is different from it. The box product can be useful for constructing counterexamples, but otherwise we're not really going to talk about it. $\square$

2.2. Metric spaces.

You may have seen metric spaces in your real analysis class, but we'll redo some of the basics from scratch. We'll skip a lot of what's in Munkres' text since it has more to do with analysis than topology, but I at least want to explain why what we're talking about is compatible with what you have learned previously. Let $\mathbf{R}_{\geq 0}$ denote the set of non-negative real numbers.

Definition 2.2.1. Let X be a set and let $d \colon X \times X \to \mathbf{R}_{\geq 0}$ be a function. We say that d is a **metric** if the following 3 conditions hold:

(1) For all $x, y \in X$, we have $d(x, y) = 0$ if and only if $x = y$.
(2) (Symmetry) For all $x, y \in X$, we have $d(x, y) = d(y, x)$.

(3) (Triangle inequality) For all $x, y, z \in X$, we have $d(x,y) + d(y,z) \geq d(x,z)$.

The pair (X, d) is called a **metric space**. ☐

Example 2.2.2. (1) For $X = \mathbf{R}^n$, the Euclidean metric is defined by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}.$$

It takes some work to show that this satisfies the triangle inequality, but we will skip that (it should have been covered in Math 140/142).

For $n = 1$, this can be simplified to $d(x, y) = |x - y|$.

(2) Let X be any set and define $d(x, y) = 1$ for all $x \neq y$ and $d(x, x) = 0$ for all x. This defines a metric, as is easy to check directly.

(3) (p-adic metric) Fix a prime number p. Given a nonzero integer x, define $v_p(x)$ to be the exponent of the largest power of p that divides x. We get a metric d on the set of integers $\mathbf{Z}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ p^{-v_p(x-y)} & \text{else} \end{cases}.$$ ☐

Definition 2.2.3. Let (X, d) be a metric space. Given $x \in X$ and $\epsilon > 0$, we define the ϵ-**ball centered at** x by

$$B_d(x, \epsilon) = \{y \in X \mid d(x, y) < \epsilon\}.$$

We will just write $B(x, \epsilon)$ if there is no ambiguity about what the metric d is. ☐

Lemma 2.2.4. *The set* $\{B_d(x, \epsilon) \mid x \in X,\ \epsilon > 0\}$ *is a basis for a topology.*

Let's explain the intuition behind the formula you're about to see. We need to do this: given balls $B(x_1, \epsilon_1)$ and $B(x_2, \epsilon_2)$ and y in their intersection, we need to find $\delta > 0$ such that

$$B(y, \delta) \subseteq B(x_1, \epsilon_1) \cap B(x_2, \epsilon_2).$$

The key idea is to use the triangle inequality: if $z \in B(y, \delta)$, we have for $i = 1, 2$:

$$d(z, x_i) \leq d(z, y) + d(y, x_i) < \delta + d(y, x_i).$$

We'd be done if we know how to choose δ so that the last quantity is less than ϵ_i. There's a lot of ways to do this, but we'll take $\delta = \min(\epsilon_1 - d(y, x_1), \epsilon_2 - d(y, x_2))$.

Proof. We need to check the definition. Pick two balls $B(x_1, \epsilon_1)$ and $B(x_2, \epsilon_2)$ and suppose there is a point y in their intersection. Set $\delta = \min(\epsilon_1 - d(y, x_1), \epsilon_2 - d(y, x_2))$, which is positive. The argument above shows that $B(y, \delta) \subseteq B(x_1, \epsilon_1) \cap B(x_2, \epsilon_2)$, so we're done. ☐

Remark 2.2.5. The "intuition" part above could have been integrated directly into the proof to make the overall text shorter. Namely, we could have immediately just given the definition of δ before understanding why and then explain why it works. We have separated it here for instructive purposes. When writing your own proofs you can do it either way. ☐

Definition 2.2.6. The topology coming from ϵ-balls is called the **metric topology** on X.

In other words: a subset $U \subseteq X$ is open if and only if for all $x \in U$, there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. ☐

Keep in mind that X can have many different metrics, and different metrics can give different topologies, so the notation should technically still refer to d.

Example 2.2.7.
(1) When $X = \mathbf{R}^n$, we've already introduced this definition where d is the Euclidean metric.
(2) If X is a finite set with a metric, then the metric topology is always the discrete topology (why?).
(3) If $d(x, y) = 1$ for all $x, y \in X$ with $x \neq y$, again the metric topology is the discrete topology. □

In any case, if we have a metric space (X, d), then we will always automatically think of this as a topological space with the metric topology unless stated otherwise.

Remark 2.2.8. You may have seen the definition of a continuous function between metric spaces before using δ-ϵ: given metric spaces (X, d_X) and (Y, d_Y), a function $f\colon X \to Y$ is continuous at $x \in X$ if for all $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_{d_X}(x, \delta)) \subseteq B_{d_Y}(f(x), \epsilon)$. This is equivalent to f being continuous in our sense using the metric topology: the proof we gave for $X = Y = \mathbf{R}$ in Example 1.7.7 carries over with some small changes. □

We should check how this interacts with other constructions: subspaces and products.

For subspaces: if X has a metric d, and $A \subseteq X$ is a subset, then d also defines a metric on A, which we'll denote by d_A for the next result.

Proposition 2.2.9. *Let $A \subseteq X$ be a subset of a metric space. The metric topology on A given by d_A is the same as the subspace topology coming from the metric topology on X.*

Proof. The main point is that $B_{d_A}(x, \epsilon) = B_d(x, \epsilon) \cap A$ for any $x \in A$. I'll leave the rest to you. □

How about products? For simplicity, we'll just discuss products of finitely many metric spaces, but see Munkres for how to deal with arbitrary products.

Definition 2.2.10. Let (X_1, d_1) and (X_2, d_2) be metric spaces. The **sup metric** d on $X_1 \times X_2$ is defined by

$$d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)).$$

More generally, if we have n metric spaces $(X_1, d_1), \dots, (X_n, d_n)$, we can define the sup metric on $X_1 \times \cdots \times X_n$ by

$$d((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max(d_1(x_1, y_1), \dots, d_n(x_n, y_n)).$$ □

Proposition 2.2.11. *The sup metric on $X_1 \times \cdots \times X_n$ is a metric.*

Proof. The first two properties are straightforward, so let's focus on the triangle inequality. Pick points $x_i, y_i, z_i \in X_i$ for $i = 1, \dots, n$. For each $i = 1, \dots, n$, since d_i is a metric, we have

$$\begin{aligned} d_i(x_i, z_i) &\le d_i(x_i, y_i) + d_i(y_i, z_i) \\ &\le \max(d_1(x_1, y_1), \dots, d_n(x_n, y_n)) + \max(d_1(y_1, z_1), \dots, d_n(y_n, z_n)) \\ &= d((x_1, \dots, x_n), (y_1, \dots, y_n)) + d((y_1, \dots, y_n), (z_1, \dots, z_n)) \end{aligned}$$

Since this is true for all i, we conclude that

$$\max(d_1(x_1, z_1), \dots, d_n(x_n, z_n)) \le d((x_1, \dots, x_n), (y_1, \dots, y_n)) + d((y_1, \dots, y_n), (z_1, \dots, z_n)),$$

which is what we want to show because the left side is $d((x_1, \dots, x_n), (z_1, \dots, z_n))$. □

Example 2.2.12. If (X_i, d_i) is $\mathbf{R}$ with the Euclidean metric for $i = 1, \ldots, n$, then the sup metric on $\mathbf{R}^n$ is

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max(|x_1 - y_1|, \ldots, |x_n - y_n|).$$

This is also called the **square metric** on $\mathbf{R}^n$: for $n = 2$, the open balls are squares, and for higher n, they are n-dimensional cubes. □

Proposition 2.2.13. *Let $X_1, \ldots, X_n$ be metric spaces. The metric topology on $X_1 \times \cdots \times X_n$ coming from the sup metric is the same as the product topology.*

Proof. From Proposition 2.1.4, we have the following basis for the product topology: subsets of the form $U = U_1 \times \cdots \times U_n$ where $U_i \subseteq X_i$ is an open set. We first show that these are open in the topology coming from the sup metric. By definition, given $x_i \in U_i$ for each i, there exists $\epsilon_i > 0$ such that $B_{d_i}(x_i, \epsilon_i) \subseteq U_i$. Now let $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Then $B_d((x_1, \ldots, x_n), \epsilon) \subseteq U$ (exercise: check this). So this proves that U is open in the sup metric topology and hence the sup metric topology refines the product topology.

To prove that they are equal, we need to also show that the product topology refines the sup metric topology. The latter has a basis of open balls: $B_d((x_1, \ldots, x_n), \epsilon)$ for $\epsilon > 0$ and $x_i \in X_i$. We will show that every open ball is also open in the product topology. This follows from the following identity:

$$B_d((x_1, \ldots, x_n), \epsilon) = \pi_1^{-1}(B_{d_1}(x_1, \epsilon)) \cap \cdots \cap \pi_n^{-1}(B_{d_n}(x_n, \epsilon))$$

because both sides are the set of tuples $(y_1, \ldots, y_n)$ such that $d_i(y_i, x_i) < \epsilon$ (exercise: check this). By definition, each $\pi_i^{-1}(B_{d_i}(x_i, \epsilon))$ is open in the product topology, and we're just taking a finite intersection of open sets, so $B_d((x_1, \ldots, x_n), \epsilon)$ is also open. □

Next, two different metrics on X can actually give the same topology.

Proposition 2.2.14. *Let d and d' be metrics on X, and let $\mathcal{T}$ and $\mathcal{T}'$ be the corresponding metric topologies. Then $\mathcal{T}'$ refines $\mathcal{T}$ if and only if the following condition holds:*

$(*)$ *For all $x \in X$, for all $\epsilon > 0$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \epsilon)$.*

Proof. First suppose that $\mathcal{T}'$ refines $\mathcal{T}$. By definition, every open set in $\mathcal{T}$ is also open in $\mathcal{T}'$. In particular, for any $x \in X$ and $\epsilon > 0$, we have that $B_d(x, \epsilon)$ is open in $\mathcal{T}$ by definition, and hence also open in $\mathcal{T}'$. But again by definition, that means that for every $y \in B_d(x, \epsilon)$, there is an open ball (with respect to d') centered at y that is contained in $B_d(x, \epsilon)$. Now apply that to $y = x$ and we see that the condition $(*)$ is true.

Conversely, suppose that $(*)$ is true. Let U be an open set in $\mathcal{T}$. To show that it is open in $\mathcal{T}'$, we need to say that for all $x \in U$, there exists $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq U$. So pick $x \in U$. Since U is open in $\mathcal{T}$, there exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$. The existence of δ is now guaranteed by $(*)$. □

Corollary 2.2.15. *The Euclidean metric and square metric on $\mathbf{R}^n$ give the same topology, which is also the product topology (using that $\mathbf{R}^n = \mathbf{R} \times \cdots \times \mathbf{R}$).*

Proof. Let d be the Euclidean metric, and temporarily let ρ be the square metric.

To show that their metric topologies are the same, it is the same to say that the Euclidean metric topology and the square metric topology mutually refine each other. So we need to verify condition $(*)$ from the previous result in both directions. This follows from the following identity:

$$B_\rho\left((x_1,\dots,x_n),\frac{\epsilon}{\sqrt{n}}\right)\subseteq B_d((x_1,\dots,x_n),\epsilon)\subseteq B_\rho((x_1,\dots,x_n),\epsilon).$$

The second inclusion is easy to see directly, and the first one says that a cube with radius $\epsilon/\sqrt{n}$ fits inside of a sphere of radius ϵ (exercise: check this).

Finally, the square metric topology is the same as the product topology by Proposition 2.2.13. ☐

That was a lot of work, but the main point is that we have confirmed that all of the topological concepts encountered in 140/142 are special cases of the definitions in this class.

Finally, one more definition.

Definition 2.2.16. A topological space X is **metrizable** if there exists a metric d such that the topology on X is the metric topology from d. ☐

In other words, metric spaces give us topological spaces, and whatever we get from that process is called metrizable. We know that not everything is metrizable (for example, any finite topological space with something that is not the discrete topology). Metrizability is a "topological property": if $X \cong Y$, then X is metrizable if and only if Y is metrizable. If we have time, we'll discuss some theorems at the end of the course about what properties of a topological space imply that it is metrizable.

2.3. **Hausdorff spaces.** This is not a construction, but rather a particular property that topological spaces might possess and is a natural topic to follow metric spaces, so we discuss it now.

Definition 2.3.1. A topological space X is **Hausdorff** if, for every pair of points x, y with $x \neq y$, there exists a neighborhood U of x and a neighborhood V of y such that $U \cap V = \varnothing$. ☐

In other words, in a Hausdorff space, we can always "separate" two different points with neighborhoods that don't intersect.

As with metrizability, being Hausdorff is a topological property.

Remark 2.3.2. These spaces are named in honor of Felix Hausdorff. According to Wikipedia, Hausdorff's original definition of a topological space included this property as an axiom. It's a good property, but given the wide uses of topology in mathematics (including spaces that fail to have this property), there are plenty of reasons not to require it in the definition. ☐

Let's knock out a few properties related to our previous definitions.

Proposition 2.3.3. *Every metrizable space X is Hausdorff.*

Proof. Suppose X has the metric topology from a metric d. Pick distinct points x, y and let $\epsilon = d(x, y)$. Set $U = B(x, \epsilon/2)$ and $V = B(y, \epsilon/2)$, which are open since $\epsilon/2 > 0$. I claim that $U \cap V = \varnothing$: if not, pick $z \in U \cap V$. Then

$$\epsilon = d(x,y) \leq d(x,z) + d(z,y) < \epsilon/2 + \epsilon/2 = \epsilon,$$

which is a contradiction. ☐

Proposition 2.3.4. *Every subspace A of a Hausdorff space X is also Hausdorff.*

Proof. Pick distinct $x, y \in A$. Since X is Hausdorff, there exist neighborhoods U, V of x, y in X that don't intersect. But then $U \cap A$ and $V \cap A$ are neighborhoods of x, y in A that again don't intersect. $\square$

Proposition 2.3.5. *Every product of Hausdorff spaces is Hausdorff.*

Proof. Let I be an index set and suppose we are given a Hausdorff space X_i for each $i \in I$. Pick two distinct tuples (x_i) and (y_i) in $\prod_{i \in I} X_i$. By definition, there is some index $j \in I$ such that $x_j \neq y_j$. Since X_j is Hausdorff, we can find neighborhoods U and V of x_j and y_j, respectively, that don't intersect. But then $\pi_j^{-1}(U)$ and $\pi_j^{-1}(V)$ are neighborhoods of (x_i) and (y_i), respectively, and they don't intersect. $\square$

Remark 2.3.6. That covers a lot of ground for us already: $\mathbf{R}^n$ is metrizable and hence Hausdorff, and so any subspace of $\mathbf{R}^n$ is also metrizable and also Hausdorff. It's natural to wonder if there are Hausdorff spaces which are not metrizable. In fact, they do exist. Depending on how far we get, we'll see other stronger properties that metrizability implies. Still, constructing a Hausdorff space which is not metrizable (with proof) involves some work, so we won't think too hard about it right now. $\square$

Remark 2.3.7. Having more open sets makes it more likely that a space is Hausdorff. More formally, if X is a set and $\mathcal{T}$ is a Hausdorff topology, then any other topology that refines $\mathcal{T}$ is also Hausdorff.

An extreme example of a non-Hausdorff space is the indiscrete topology (assuming that X has more than 1 point). $\square$

This is also a good place to discuss sequential limits.

Definition 2.3.8. Let $x_1, x_2, \ldots$ be a sequence of elements of a topological space X. Given $x \in X$, the sequence **converges to** x if, for every neighborhood U of x, there exists a positive integer N such that $x_n \in U$ for all $n \geq N$ (in other words, every neighborhood of x must contain the sequence except for finitely many exceptions). $\square$

Make sure you understand the order of quantifiers here: the value N is allowed to depend on the neighborhood, so there is no requirement that a single value of N works for every neighborhood at the same time.

Proposition 2.3.9. *If $x_1, x_2, \ldots$ belong to a subset A and converges to x, then $x \in \overline{A}$.*

Proof. By definition, every neighborhood of x contains some x_i, so intersects A. $\square$

In general, the converse need not hold, i.e., if $x \in \overline{A}$, there might not be a sequence of elements of A converging to x. But it does hold for metrizable spaces.

Proposition 2.3.10. *If X is metrizable, and $A \subseteq X$ is a subset, then for any $x \in \overline{A}$, there exists a sequence $x_1, x_2, \cdots \in A$ that converges to x.*

Proof. Let d be a metric giving the topology on X. Since $x \in \overline{A}$, for each positive integer n, the neighborhood $B_d(x, 1/n)$ intersects A; let x_n be any point inside that intersection. We claim that $x_1, x_2, \ldots$ converges to x. So let U be any neighborhood of x. There exists $\epsilon > 0$ such that $B_d(x, \epsilon) \subseteq U$. So if $1/n < \epsilon$, then $x_n \in U$. So take N in the definition to be any integer such that $N > 1/\epsilon$. $\square$

If you unpackage this definition for $X = \mathbf{R}$, you get the usual δ-ϵ definition of convergence. But be warned: some very strange behavior can happen for general topological spaces. First, a sequence might not converge to anything (this should not be surprising, this already happens in $\mathbf{R}$). Second, a sequence might converge to multiple different points!

Example 2.3.11. Let X be any set with the indiscrete topology. Then every sequence converges to every point. $\square$

Proposition 2.3.12. *If X is Hausdorff, then every sequence converges to at most one point.*

Proof. Let $x_1, x_2, \dots$ be a sequence and assume that there are two different points x and y which the sequence converges to. Since X is Hausdorff, we can find neighborhoods U and V of x and y, respectively, that don't intersect. By definition of convergence, there exists N such that $x_n \in U$ for $n \geq N$, and similarly, there exists N' such that $x_n \in V$ for $n \geq N'$. But then $x_m \in U \cap V$ for $m = \max(N, N')$, which is a contradiction. $\square$

2.4. **Quotient spaces.** First, let's remind ourselves what an equivalence relation is.

Definition 2.4.1. Let X be a set. An **equivalence relation** on X is a subset $R \subseteq X \times X$ (the notation is $x \sim y$ if $(x, y) \in R$) such that:

(1) For all $x \in X$, we have $x \sim x$.
(2) (Symmetric) If $x \sim y$, then $y \sim x$.
(3) (Transitive) For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x \sim z$.

We use the phrase "$\sim$ is an equivalence relation on X" and we write $x \not\sim y$ if $(x, y) \notin R$.

A subset $S \subseteq X$ is an **equivalence class** if:

(1) If $x, y \in S$, then $x \sim y$.
(2) If $x \in S$ and $y \notin S$, then $x \not\sim y$.

In words, an equivalence class is a subset of elements which are all equivalent to each other, and contains all of the elements that are equivalent to any of its members.

Every element belongs to an equivalence class, and $X/{\sim}$ is notation for the set of equivalence classes. For $x \in X$, we use $[x]$ to mean the equivalence class that contains x. There is a surjective function $\pi \colon X \to X/{\sim}$ given by $\pi(x) = [x]$, which we call the **quotient map**. $\square$

Remark 2.4.2. When specifying an equivalence relation, we generally need to specify all pairs that are equivalent, but it can get tedious to have to say things like $x \sim x$ for all x (because it must always be true) and also to specify consequences of the transitive property, i.e., if we say that $x \sim y$ and $y \sim z$, it is understood that we don't also need to say that $x \sim z$ because it's forced. So we'll usually just specify the interesting equivalences and omit the forced ones. $\square$

Example 2.4.3. If $X = \mathbf{Z}$, declaring that $x \sim y$ if they have the same parity (both odd or both even) is an equivalence relation, and it has two equivalence classes: the set of even integers and the set of odd integers, and $\mathbf{Z}/{\sim}$ is also usually called $\mathbf{Z}/2$. Then $\pi(x)$ is the set of even integers if x is even and otherwise $\pi(x)$ is the set of odd integers. $\square$

Now we come to the setting where X has a topology and an equivalence relation. Then we say that a subset $U \subseteq X/{\sim}$ is open if $\pi^{-1}(U)$ is open in X. We call this the **quotient topology** on $X/{\sim}$ and it is the default topology that we use unless otherwise specified.

Proposition 2.4.4. *The quotient topology is a topology.*

The proof is left as a homework problem.

Of course, the quotient map $\pi\colon X \to X/\!\sim$ is, by definition, continuous.

The quotient topology has a very important property. Suppose $f\colon X \to Y$ is a continuous function such that f is constant on equivalence classes of $\sim$, i.e., if $x \sim x'$, then $f(x) = f(x')$. Then we can define a function $g\colon X/\!\sim \to Y$ where for any equivalence class S, $g(S)$ is defined to be $f(x)$ for any $x \in S$ (by assumption, it does not matter which element of S is picked).

Proposition 2.4.5. *$g\colon X/\!\sim \to Y$ is continuous.*

Proof. First, note that $f = g \circ \pi$ by definition. Let $U \subseteq Y$ be an open subset. Then $\pi^{-1}(g^{-1}(U)) = f^{-1}(U)$, and $f^{-1}(U)$ is open in X since f is continuous. But then by the definition of quotient topology, we see that $g^{-1}(U)$ is open in $X/\!\sim$. □

Remark 2.4.6. The quotient topology is in fact the unique topology on $X/\!\sim$ such that the quotient map is continuous, and the above result holds. Compare this to the situation of the product topology (it solves an optimization problem of having just enough open sets). □

This actually gives us a lot of new examples and new ways to view spaces from before.

Example 2.4.7. Consider $X = [0, 1]$ with the usual subspace topology from $\mathbf{R}$. Define $0 \sim 1$ (remember, we're omitting spelling out the forced things like $x \sim x$, so I won't say that next time). I claim that $X/\!\sim$ is homeomorphic to the circle S^1.

First, as in Example 1.7.10 we have a continuous function $f\colon X \to \mathbf{R}^2$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$ whose image lies in S^1, so we can replace $\mathbf{R}^2$ with S^1. Also, f is constant on equivalence classes, so we have a continuous function $g\colon X/\!\sim \to \mathrm{S}^1$. We also know from trigonometry that g is a bijection. To show that g is a homeomorphism, we need to show that if $U \subseteq X/\!\sim$ is open then so is $g(U)$.

To do that rigorously, let's consider two separate cases. If U does not contain the equivalence class $\{0, 1\}$, then U consists of singleton equivalence classes, and $\pi^{-1}(U)$ is an open set in $[0, 1]$ that does not contain 0 or 1, and hence is a union of open intervals. The image of an open interval in S^1 is an arc (minus its end points) which is open in S^1 (draw picture here), so the image of U is a union of open arcs and hence open. If U contains $\{0, 1\}$, then the only difference is that now $\pi^{-1}(U)$ contains both 0 and 1, so is a union of open intervals together with the half-intervals $[0, a)$ and $(b, 1]$ for some $a, b \in (0, 1)$. But $f([0, a) \cup (b, 1])$ is also an open arc (draw picture here), so this case is also ok. □

That last verification that g^{-1} is continuous was annoyingly complicated. We'll see later how it can be simplified once we talk about compact spaces.

Example 2.4.8. If X is Hausdorff, then is $X/\!\sim$ also Hausdorff? Unfortunately, this can fail. Here's an example we can try to visualize. Start with $X = \mathbf{R}^2$ and declare that $x \sim y$ if x is a nonzero multiple of y. Then $\{(0, 0)\}$ is an equivalence class, as is every line through the origin with $(0, 0)$ removed. But then every neighborhood of the origin intersects every other equivalence class, so there is no way to separate it from another equivalence class.

What about conditions that guarantee that $X/\!\sim$ is Hausdorff? There are various theorems of this form, but probably too complicated to go into, so we'll just settle for studying special examples. □

Let's finish with a few more examples.

Example 2.4.9. Let $n \geq 0$ be an integer and start with $X = \mathbf{R}^{n+1} \setminus \{(0, \ldots, 0)\}$, thought of as a subspace of $\mathbf{R}^{n+1}$. Define two points to be equivalent if one is a (nonzero) scalar multiple of the other. The quotient space $X/\sim$ is called **real projective space** and is usually denoted by $\mathbf{RP}^n$ or $\mathbf{P}^n_{\mathbf{R}}$. This is a Hausdorff space, but I'll leave the details for you.

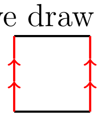
If $n = 0$, this is just a single point, so there's no mystery about its topology.

If $n = 1$, then $\mathbf{RP}^1$ turns out to be homeomorphic to the circle S^1. This is a little involved, so let's skip the details for now.

For $n \geq 2$, these spaces are new examples for us. $\mathbf{RP}^2$ is also called the **real projective plane**. It can't be embedded into $\mathbf{R}^3$ so it's hard to think about or draw, though look up "Boy's surface" for an approximation. $\square$

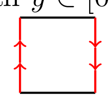
Example 2.4.10. Now let's consider $X = [0, 1] \times [0, 1]$, a unit length square. There are some interesting quotient spaces we can take by identifying points on the boundary.

(1) Tube ($\mathrm{S}^1 \times [0, 1]$). For this one, we have $(0, y) \sim (1, y)$ for all $y \in [0, 1]$. Pictorially, we draw this as the following diagram:



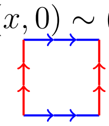
If you take a (square) sheet of paper and tape together one pair of opposite sides, that's the space you get.

(2) Möbius band. For this identify the sides in an opposite way: $(0, y) \sim (1, 1 - y)$ for all $y \in [0, 1]$:



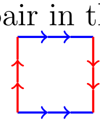
Again, take a sheet of paper and tape together one pair of opposite sides, but first twist it once. This is a surface with only "one side".

(3) Torus. Now let's identify both pairs of opposite sides: $(0, y) \sim (1, y)$ for all y and $(x, 0) \sim (x, 1)$ for all x:



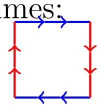
Start with the tube in example (1) and now tape together the circles on both sides without any twisting.

(4) Klein bottle. Identify one pair of sides the normal way and then identify the other pair in the opposite way: $(1, 1 - y) \sim (0, y)$ for all y and $(x, 0) \sim (x, 1)$ for all x:



This one you cannot physically do: you would have to take the ring in (1) and identify the opposite circles in a completely reversed way. The problem is that this quotient space cannot be embedded in $\mathbf{R}^3$ (but it can be embedded in $\mathbf{R}^4$). We won't prove that though.

(5) Real projective plane. The last variation is to identify sides in the reversed way both times:



We won't prove that this is homeomorphic to $\mathbf{RP}^2$, but I just wanted to say it for completeness. $\square$

There's another common way that quotient spaces are used, and that is to "glue" together two topological spaces X and Y along a common subspace. First, the disjoint union $X \amalg Y$ has the following topology: open sets are the subsets of the form $U \amalg V$ for open subsets $U \subseteq X$ and $V \subseteq Y$. Let $A \subseteq X$ be a subspace and suppose we are given a continuous function $f \colon A \to Y$. Then we define $\sim$ on $X \amalg Y$ by $a \sim f(a)$ for all $a \in A$ (plus everything else we need to do: $f(a) \sim a$ and identifying everything with itself). The intuition is that we are gluing X onto Y along A in a way prescribed by f.

Example 2.4.11. Take $X = Y = \mathrm{S}^1$ and take $A = \{(0,1)\}$ a single point. Define $f(0,1)$ to be any point (it doesn't matter, but for concreteness, take $(0,1)$ again). The quotient space can be drawn as an infinity symbol ∞ (first we'd have to embed this into $\mathbf{R}^2$ but let's skip the formula). □

Remark 2.4.12. This construction is crucial for the definition of a "CW complex" which is an important object in algebraic topology. In that case, we're building spaces by gluing balls onto existing spaces along its sphere boundary (so $X = \mathrm{B}^n$ and $A = \mathrm{S}^{n-1}$). □

3. Connected spaces

3.1. Definitions and first properties.

Definition 3.1.1. Let X be a (nonempty) topological space. A **separator** of X is a pair of nonempty open subsets U and V such that $U \cap V = \varnothing$ and $U \cup V = X$. If X has a separator, we say that X is **disconnected**. Otherwise, we say that X is **connected**. □

Alternatively, X is connected if and only if its only clopen subsets are $\varnothing$ and X.

Being connected is a topological property: if $X \cong Y$ then X is connected if and only if Y is connected.

Remark 3.1.2. Given how "connected" is defined (the non-existence of something), it will usually be most natural to prove that a space is connected by assuming that it is not and arriving at a contradiction. □

Remark 3.1.3. Technically, the empty set can be considered a topological space. Should it be considered connected? There seem to be good reasons for both possible conventions. This won't be an issue for us though, so we don't need to address it.

Below, when referencing connected spaces, we'll always assume they are nonempty even if we forget to specify. □

Example 3.1.4.
(1) Let $X = \{1,2\}$. If it has the discrete topology, then $\{1\}$ and $\{2\}$ form a separator, so X is disconnected. If it has the indiscrete topology, then there is no separator, so X is connected.
(2) If $X = [0,1] \cup [2,3]$, a subset of $\mathbf{R}$, then $[0,1]$ and $[2,3]$ give a separator, so X is disconnected. We will prove soon that intervals are always connected.
(3) Consider the rational numbers $\mathbf{Q}$ as a subset of $\mathbf{R}$. Then any subset A of $\mathbf{Q}$ with at least 2 elements is disconnected. For example, if $x, y \in A$ and $x < y$, then we can always find an irrational number r such that $x < r < y$, so $A \cap (-\infty, r)$ and $A \cap (r, \infty)$ form a separator for A. Each is nonempty because the first contains x and the second contains y. □

Definition 3.1.5. Let $A \subseteq \mathbf{R}$ be a nonempty set of real numbers. We say that A is **convex** if, given $a, b \in A$ such that $a < b$, we have $[a,b] \subseteq A$. □

Example 3.1.6. Any interval (with either side open or closed, and allowed to be $\pm\infty$) is convex. In particular, $\mathbf{R}$ is convex. $\square$

The following result is very important.

Theorem 3.1.7. *A (nonempty) subspace A of $\mathbf{R}$ is connected if and only if it is convex.*

Proof. First, assume that A is not convex. Then there exist $a < c < b$ with $a, b \in A$ but $c \notin A$. Then $U = (-\infty, c) \cap A$ and $V = (c, \infty) \cap A$ is a separator for A: both are nonempty because $a \in U$ and $b \in V$, they clearly do not intersect, and their union is all of A because $c \notin A$. Hence A is disconnected.

Now assume that A is convex. We prove that A is connected by contradiction. So assume there is a separator U', V' for A. Pick $a \in U'$ and $b \in V'$. Without loss of generality, we can assume that $a < b$ (if not, just rename U' and V'). Since A is convex, $[a, b] \subseteq A$ and so $U = U' \cap [a, b]$ and $V = V' \cap [a, b]$ give a separator for $[a, b]$.

Let $c = \sup U$, which recall is the least upper bound of all of the elements in U. From real analysis, we know both that c exists and $c \in [a, b]$. Then either $c \in U$ or $c \in V$, and we consider both cases separately.

Case 1. Suppose that $c \in U$. Since U is open in $[a, b]$, there exists $\epsilon > 0$ such that $(c-\epsilon, c+\epsilon) \cap [a, b] \subseteq U$, and since $b \notin U$, we know that $b \notin (c-\epsilon, c+\epsilon)$ and so $[c, c+\epsilon) \subseteq U$. But $c + \epsilon/2 > c$, which contradicts the fact that c is an upper bound for U.

Case 2. Suppose that $c \in V$. As before, since V is open in $[a, b]$, there exists $\epsilon > 0$ so that $(c-\epsilon, c+\epsilon) \cap [a, b] \subseteq V$. Again, since $a \notin V$, we have $a \notin (c-\epsilon, c+\epsilon)$, and so $(c-\epsilon, c] \subseteq V$. But then $c - \epsilon/2$ must also be an upper bound for U, which is again a contradiction.

Since both possibilities lead to a contradiction, we must conclude that a separator does not exist, and hence A is connected. $\square$

We're now going to examine how the connected property interacts with our previous definitions.

Proposition 3.1.8. *Let $f \colon X \to Y$ be a continuous function. If X is connected, then so is the image $f(X)$.*

Proof. We will prove the contrapositive. Assume that $f(X)$ is disconnected; we will prove that X is also disconnected. Then we can find a separator U, V for $f(X)$. By definition of subspace topology, there exist open sets U', V' of Y such that $U = U' \cap f(X)$ and $V = V' \cap f(X)$.

Now define $A = f^{-1}(U')$ and $B = f^{-1}(V')$, which we claim is a separator for X. First, they are open because f is continuous. Next, they are nonempty: since U is nonempty and is contained in the image of f, there exists $x \in X$ such that $f(x) \in U$; then $x \in A$, and similarly for B. Finally, $A \cup B = X$: for all $x \in X$, we have $f(x) \in U \cup V$ since their union is $f(X)$, and so $x \in A \cup B$. $\square$

Corollary 3.1.9. *Let $f \colon X \to Y$ be a continuous function. If $A \subseteq X$ is a subspace which is connected, then $f(A)$ is connected.*

Proof. Apply the previous result to the continuous function $f|_A \colon A \to Y$. $\square$

Corollary 3.1.10. *If X is connected and $\sim$ is an equivalence relation, then the quotient space $X/{\sim}$ is connected.*

Proof. We have a surjective continuous function $\pi\colon X \to X/{\sim}$. □

We'll mention some examples later when we have more results.

Remark 3.1.11. What about preimages of connected spaces? Take $X = [0, 1] \cup [2, 3]$ and take $Y = \{0\}$. The (only) function $f\colon X \to Y$ is continuous and Y is connected, but $f^{-1}(Y)$ is disconnected. So we can't draw any general conclusions about preimages without further information. □

Before continuing, let's combine the above to prove a familiar result in calculus:

Theorem 3.1.12 (Intermediate value theorem)**.** *Given $a < b \in \mathbf{R}$, let $f\colon [a, b] \to \mathbf{R}$ be a continuous function. For all $x, y \in [a, b]$, if $r \in \mathbf{R}$ satisfies $f(x) < r < f(y)$, then there exists $c \in [a, b]$ such that $f(c) = r$.*

Proof. The conclusion just says that the image is convex. But we know that is true because $[a, b]$ is connected, which implies that its image is connected, and we just proved that's the same as being convex (for subsets of $\mathbf{R}$). □

Knowing how the proof works, you can see that the domain being a closed interval isn't crucial: it can be any convex subset of $\mathbf{R}$. But actually it's not really more general to do that since once you have a convex set, you can always restrict the function to $[x, y]$ to reduce to the above statement.

Example 3.1.13. The intermediate value theorem has a lot of applications in calculus, but that is better suited for another course. Instead, as an application, let's solve this exercise from Munkres: if $f\colon \mathrm{S}^1 \to \mathbf{R}$ is continuous, then there exists $x \in \mathrm{S}^1$ such that $f(x) = f(-x)$ ($-x$ just comes from thinking of S^1 as a subset of $\mathbf{R}^2$). Alternatively, there is a pair of antipodal points that have the same image. Define a function $g\colon \mathrm{S}^1 \to \mathbf{R}$ by $g(x) = f(x) - f(-x)$. It is equivalent to show that 0 is in the image of g.

First, since S^1 is connected (for example, it's a quotient of an interval as shown in Example 2.4.7), the image of g is a convex set. Now pick a point $x \in \mathrm{S}^1$. If $f(x) = f(-x)$, we're done. Otherwise, $g(x)$ is either positive or negative. But $g(-x)$ has the opposite property. Now by convexity, 0 must be in the image. □

Let's continue with more properties.

Lemma 3.1.14. *Let X be a disconnected space and let U, V be a separator for X. If $Y \subseteq X$ is a connected subspace, then we must have either $Y \subseteq U$ or $Y \subseteq V$.*

Proof. Consider the subsets $Y \cap U$ and $Y \cap V$. They are open, don't intersect, and their union is all of Y. Hence if both are nonempty, they would be a separator for Y. But since Y is connected, this is not possible, so one of them must be empty. If $Y \cap U = \varnothing$, then since $U \cup V = X$, we must have $Y \subseteq V$. And if $Y \cap V = \varnothing$, then we have $Y \subseteq U$. □

Proposition 3.1.15. *Let X be a topological space and let A be a connected subspace. If $B \subseteq X$ is a subspace such that $A \subseteq B \subseteq \overline{A}$, then B is also connected.*

Proof. Suppose for contradiction that B has a separator U, V. Since A is connected, we have either $A \subseteq U$ or $A \subseteq V$ by Lemma 3.1.14. Without loss of generality, we can assume that $A \subseteq U$ (rename U and V otherwise). But then $\overline{A} \subseteq \overline{U}$, and also $\overline{U} \subseteq \overline{B} \subseteq \overline{A}$ which means that $\overline{U} = \overline{B}$ and hence $B = \overline{U} \cap B$. By Corollary 1.6.6, we have $\overline{U} \cap B = \mathrm{Cl}_B(U)$. Finally, U is closed relative to B (because V is open relative to B), so we have $\mathrm{Cl}_B(U) = U$. In conclusion, $B = U$, which contradicts that V is nonempty. □

In particular, the closure of a connected subspace is connected, i.e., adding all of the limit points does not destroy the connected property. But more generally, adding only some of the limit points also preserves the connected property.

Remark 3.1.16. The corresponding statement with interiors can fail. For example, take two closed balls in $\mathbf{R}^2$ that intersect in exactly one point. Then their interior is a union of open balls that don't intersect, and this is disconnected. $\square$

Proposition 3.1.17. *Let X be a topological space and let I be an index set. Suppose for each $i \in I$, we have a connected subspace $A_i \subseteq X$. If $\bigcap_{i\in I} A_i \neq \varnothing$, then $\bigcup_{i\in I} A_i$ is also connected.*

Proof. We prove this by contradiction, so assume that $\bigcup_{i\in I} A_i$ has a separator U, V. Pick a point $x \in \bigcap_{i\in I} A_i$. Then either $x \in U$ or $x \in V$; without loss of generality, we may assume that $x \in U$ (just rename U and V otherwise). By Lemma 3.1.14, since for each $i \in I$, A_i is connected, we must have either $A_i \subseteq U$ or $A_i \subseteq V$. Since $x \notin V$ and $x \in A_i$, it must be that $A_i \subseteq U$ for all i. But that implies that $\bigcup_{i\in I} A_i \subseteq U$, which contradicts that V is nonempty. $\square$

Remark 3.1.18. Of course, the assumption that the intersection is nonempty is crucial. For instance, $[0, 1]$ and $[2, 3]$ are connected subspaces of $\mathbf{R}$, but their union is not. $\square$

Finally, we consider product spaces. Let's just consider the finite case. The infinite case is more involved, so we'll leave it as an optional homework problem. First, let's handle the product of two spaces.

Proposition 3.1.19. *If X, Y are connected, then so is $X \times Y$.*

(For a visual aid for the next proof, see Figure 23.2 in Munkres' book.)

Proof. First, for any $y \in Y$, the subspace $X \times \{y\}$ is homeomorphic to X and for any $x \in X$, $\{x\} \times Y$ is homeomorphic to Y (left as exercises). In particular, both types of subspaces are connected. For $x \in X$ and $y \in Y$, define

$$T_{x,y} = (X \times \{y\}) \cup (\{x\} \times Y).$$

Since this is a union of two connected subspaces with nonempty intersection ((x, y) is in it), $T_{x,y}$ is also connected by Proposition 3.1.17.

Now pick a point $b \in Y$. Then

$$X \times Y = \bigcup_{x\in X} T_{x,b}$$

because for any $(x, y) \in X \times Y$, we have $(x, y) \in \{x\} \times Y \subseteq T_{x,b}$. But also, $X \times \{b\} \subseteq \bigcap_{x\in X} T_{x,b}$, so again we conclude from Proposition 3.1.17 that $X \times Y$ is connected. $\square$

Corollary 3.1.20. *If $X_1, \dots, X_n$ are connected topological spaces, then so is $X_1 \times \cdots \times X_n$.*

Proof. We can generalize the previous proof, but the notation is a little messy. Instead, let's just reduce to the $n = 2$ case using induction on n. If $n = 1$, there is nothing to do. Otherwise, assume we know that the product $n - 1$ connected spaces is always connected. The main observation is that $X_1 \times \cdots \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$ (left as an exercise), and the latter is a product of two connected spaces, which we know is connected by the previous result. $\square$

Example 3.1.21. All of the examples of quotient spaces we gave like the torus, Möbius band, and Klein bottle are connected since they are quotient spaces of the unit square which is the product of two intervals (and hence connected). □

Finally, as mentioned earlier, being connected is a topological property. So if one space is connected and the other is not, they cannot be homeomorphic. By itself, this is fairly limited, but there are some variations that give us new information.

If X is connected, define $x \in X$ to be a **cut point** if $X \setminus \{x\}$ is disconnected. If $f\colon X \to Y$ is a homeomorphism and $x \in X$, then x is a cut point if and only if $f(x)$ is a cut point of Y because f restricts to a homeomorphism between $X \setminus \{x\}$ and $Y \setminus \{f(x)\}$ (check this). In particular, homeomorphic spaces must have the same number of cut points (whether this is a finite number or not). If they both have infinitely many, it can also be useful to count how many non-cut points they have.

Example 3.1.22. In Example 1.8.8, we considered the function $g\colon [0, 2\pi) \to \mathrm{S}^1$ and explained why it is not a homeomorphism. In fact, these spaces are not homeomorphic because S^1 has no cut points (why?), but $[0, 2\pi)$ does have cut points (in fact, everything except 0 is a cut point). So changing the function will not help. □

You can find more examples of this idea in homework.

3.2. **Path-connected spaces.** Connectedness can be hard to check. There's a stronger condition, called "path-connected", that is easier to check (but then some connected spaces don't satisfy it) and it crucially uses the fact that closed intervals in $\mathbf{R}$ are connected, which we proved in the last section. Fortunately, for a large class of examples, such as open subspaces of $\mathbf{R}^n$, these notions turn out to be equivalent.

Definition 3.2.1. Let X be a (nonempty) topological space. Given $x, y \in X$, a **path** from x to y is a continuous function $\gamma\colon [a, b] \to X$ for some real numbers $a < b$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

We say that X is **path-connected** if, given any two points $x, y \in X$, there exists a path from x to y. □

"Path-connected" is a topological property.

For $c < d$, we know (from homework) that there is a homeomorphism $f\colon [a, b] \to [c, d]$, and we can even find one such that $f(a) = c$ and $f(b) = d$. In particular, in the definition of path, the actual choice of a and b does not really matter, since $\gamma \circ f^{-1}\colon [c, d] \to X$ is also a path from x to y. So in fact, we could have required that the domain of γ is always $[0, 1]$ and the definition would logically be equivalent. But it is useful to be flexible because it can be annoying to have to reparametrize in the middle of a definition or a proof. On the other hand, sometimes we will deal with several paths at the same time and it can be useful to force them all to have the same domain. Also, it can be useful to use $[0, 1]$ to avoid having too many letters, so we'll usually do that for convenience.

Proposition 3.2.2. *If X is path-connected, then X is connected.*

Proof. Assume not, so that there is a separator U, V for X. Pick $x \in U$ and $y \in V$. Since X is path-connected, there is a path $\gamma\colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. But then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are both open, their union is all of $[0, 1]$ since $U \cup V = X$, and are both nonempty since $0 \in \gamma^{-1}(U)$ and $1 \in \gamma^{-1}(V)$. But we already know that $[0, 1]$ is connected, so we have a contradiction. □

The converse is not true, i.e., there exist connected spaces which are not path-connected. The standard example is the "topologist's sine curve", which we'll discuss soon. We'll also want to examine if the results in the previous section hold if we replace "connected" with "path-connected". Some do, and with easier proofs, but some are false. Let's do one since it will give more information about the topologist's sine curve example and then do the rest after the example.

Proposition 3.2.3. *If $f: X \to Y$ is a continuous function and X is path-connected, then $f(X)$ is also path-connected.*

Proof. Pick two points in $f(X)$. We can write them as $f(x)$ and $f(y)$ for some $x, y \in X$. Let $\gamma: [0,1] \to X$ be a path from x to y. Then $f \circ \gamma: [0,1] \to Y$ is a path from $f(x)$ to $f(y)$. $\square$

Example 3.2.4. Consider the function $f: \mathbf{R}^n \setminus \{(0,\dots,0)\} \to \mathrm{S}^{n-1}$ given by $f(x) = x/|x|$. This is continuous and surjective. If $n \geq 2$, then the domain is path-connected (why?), so we see that S^{n-1} is also path-connected. For $n = 1$, $\mathbf{R} \setminus \{0\}$ is disconnected, and so is $\mathrm{S}^0 = \{1, -1\}$. $\square$

Example 3.2.5. Consider the continuous function $f: (0,\infty) \to \mathbf{R}^2$ given by $f(x) = (x, \sin(\frac{1}{x}))$. Let X be the image of f. Since the domain is path-connected, X is also path-connected, and in particular X is connected. The closure $\overline{X}$ is called the **topologist's sine curve**. The new points that get added are $\{(0,y) \mid -1 \leq y \leq 1\}$. I won't prove that completely, but let me explain why $(0,0) \in \overline{X}$: every neighborhood contains an ϵ-ball, and we can always find a positive integer n so that $\epsilon > 1/2\pi n$, and $f(1/2\pi n) = (1/2\pi n, 0)$ is then in the corresponding neighborhood of $(0,0)$.

In any case, $\overline{X}$ is connected because it is the closure of a connected space (Proposition 3.1.15). I claim that $\overline{X}$ is not path-connected. More specifically, there is no path that begins at a point in the y-axis and ends at a point in X.

Suppose for contradiction that there is such a path, and call it $\gamma: [a,b] \to \overline{X}$. Write the component functions of γ as $\gamma_1(t)$ and $\gamma_2(t)$ so that $\gamma(t) = (\gamma_1(t), \gamma_2(t))$. Then $\gamma_1^{-1}(0)$ is a closed subset of $[a,b]$, let c be its supremum. Since $\gamma(1) \in X$, we have $c < b$, and we now consider the path $g: [c,b] \to \overline{X}$ obtained by restricting the domain. Now reparametrize the domain so that our function is actually $g: [0,1] \to \overline{X}$ (this will make some formulas simpler), and let g_1 and g_2 be its component functions. In particular, $g_1(0) = 0$ and $g_1(t) > 0$ for all $t \in (0,1]$.

To get a contradiction, we find a sequence of points $t_1, t_2, \dots$ in $[0,1]$ such that $\lim_{n\to\infty} t_n = 0$ and $g_2(t_n) = (-1)^n$: this will show that g_2 is not continuous since $\lim_{n\to\infty} g_2(t_n)$ does not exist. To define t_n, first pick $u \in (0, g_1(1/n))$ such that $\sin(1/u) = (-1)^n$ (this essentially amounts to taking u to be the reciprocal of a large integer multiple of π). By the intermediate value theorem applied to the restriction of g_1 to $[0,1/n]$, there exists $t_n \in [0,1/n]$ such that $g_1(t_n) = u$. This is the sequence that we want, so we see that $\overline{X}$ is not path-connected.

Here are some important points that this example illustrates.

- First, a connected space can fail to be path-connected. Furthermore, $\overline{X}$ is metrizable because it is a subset of $\mathbf{R}^2$, so being connected and metrizable still does not imply path-connected.
- Second, the closure of a path-connected space can fail to be path-connected, so the analogue of Proposition 3.1.15 with "connected" replaced with "path-connected" is not valid. This fails in the worst possible way in this example: any subset Y with

$X \subsetneq Y \subseteq \overline{X}$ is also not path-connected (the argument above works as long as there is at least one point on the y-axis in Y). $\square$

Let's generalize convexity to subsets of $\mathbf{R}^n$.

Definition 3.2.6. Let A be a (nonempty) subset of $\mathbf{R}^n$. We say that A is **convex** if, given $a, b \in A$, the line segment between a and b, $\{(1-t)a + tb \mid t \in [0,1]\}$, is contained in A. $\square$

Note that for $n = 1$, this agrees with our previous definition. The next result follows from the definition.

Proposition 3.2.7. *Convex subsets of $\mathbf{R}^n$ are path-connected, and hence also connected.*

Example 3.2.8. Unlike the $n = 1$ case, connected subsets need not be convex. For example take $n = 2$ and $A = \mathbf{R}^2 \setminus \{(0,0)\}$. This is not convex (the line segment through $(1,1)$ and $(-1,1)$ goes through $(0,0)$) but it is path-connected. $\square$

Example 3.2.9. A closed or open ball is convex. For simplicity, consider the case of the open ball B centered at the origin of radius 1. If $x, y \in B$, then for $t \in [0,1]$, we have, using the triangle inequality

$$|(1-t)x + ty| \leq |(1-t)x| + |ty| = (1-t)|x| + t|y| < (1-t) + t = 1. \qquad \square$$

Finally, we consider the other results proven. Here it will be convenient to discuss concatenation of paths. Say we have 3 points $x, y, z \in X$ and that there is a path γ from x to y and a path γ' from y to z. From our discussion of reparametrizing the domain, let's just assume that the domain of γ is $[0,1]$ and the domain of γ' is $[1,2]$. Define $\gamma'' \colon [0,2] \to X$ by

$$\gamma''(t) = \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq 1 \\ \gamma'(t) & \text{if } 1 \leq t \leq 2 \end{cases}.$$

By the pasting lemma (Lemma 1.7.13), γ'' is also continuous, and gives a path from x to z. This is the **concatenation** of the paths, and we will denote it by $\gamma'' = \gamma * \gamma'$. This is one instance where it can be annoying to always use the same domain for our paths and why it is convenient to reparametrize.

We can get a much stronger version of the union of connected subspaces result:

Proposition 3.2.10. *Let X be a topological space and let I be an index set. Suppose for each $i \in I$, we have a path-connected subspace $A_i \subseteq X$. If $A_i \cap A_j \neq \varnothing$ for any $i, j \in I$, then $\bigcup_{i \in I} A_i$ is also path-connected.*

Proof. Pick $x, y \in \bigcup_{i \in I} A_i$. Then there exists $j, k \in I$ so that $x \in A_j$ and $y \in A_k$. Pick $p \in A_j \cap A_k$. Since A_j is path-connected, there is a path from x to p inside A_j (and hence also in $\bigcup_{i \in I} A_i$). Similarly, there is a path from p to y in $\bigcup_{i \in I} A_i$. Now just concatenate the paths. $\square$

Proposition 3.2.11. *Let I be an index set and let X_i be a path-connected space for all $i \in I$. Then $\prod_{i \in I} X_i$ is also path-connected.*

Proof. Pick tuples (x_i) and (y_i) in $\prod_{i \in I} X_i$. Since X_i is path-connected, there is a path γ_i from x_i to y_i. We may assume the domain is $[0,1]$. Now define $\gamma \colon [0,1] \to \prod_{i \in I} X_i$ by $\gamma(t) = (\gamma_i(t))$. Since its components are all continuous, γ is also continuous, and it goes from (x_i) to (y_i). $\square$

3.3. **Components and local properties.** Although general spaces need not be (path-)connected, we can break them into pieces, called components, which are. We handle the connected version first.

Definition 3.3.1. Let X be a (nonempty) topological space. For $x, y \in X$, define $x \sim y$ if there exists a connected subspace $A \subseteq X$ such that $x, y \in A$. We will show that $\sim$ is an equivalence relation soon, but assuming that it is, the equivalence classes of $\sim$ are called the **connected components** of X. □

Lemma 3.3.2. *$\sim$ is an equivalence relation.*

Proof. For all $x \in X$, $A = \{x\}$ is connected, so $x \sim x$.

It is clear from the definition that if $x \sim y$, then also $y \sim x$.

Now suppose $x \sim y$ and $y \sim z$. Then there exist connected subspaces A and B so that $x, y \in A$ and $y, z \in B$. But then $A \cap B \neq \varnothing$ since it contains y, so $A \cup B$ is connected (by Proposition 3.1.17) and contains x and z, so $x \sim z$. □

Proposition 3.3.3. *(1) If A is any connected subspace of X, then A is contained in some connected component.*
(2) Each connected component is a connected subspace of X.
(3) Any distinct pair of connected components don't intersect, and the union of all connected components is X.

Proof. (1) If not, then A intersects two different equivalence classes, so that there exists x, y such that $x \not\sim y$ but $x, y \in A$. But that contradicts the definition of $\sim$.

(2) Let C be a connected component of X and pick any point $x \in C$. For all $y \in C$, we have $x \sim y$ and hence there is a connected subspace A_y of X such that $x, y \in A_y$. We have $A_y \subseteq C$ by (1). Next $x \in \bigcap_{y \in C} A_y$, so this intersection is nonempty, hence $\bigcup_{y \in C} A_y$ is connected (by Proposition 3.1.17), but this union is just C.

(3) This is a general fact about equivalence classes. □

It follows from the above result that a subspace A is a connected component if it is connected, and given any other connected subspace B such that $A \subseteq B$, we must have $A = B$, i.e., A is a maximal (with respect to inclusion) connected subset.

Example 3.3.4. If $X = [0, 1] \cup [2, 3]$, then it has two connected components, which are $[0, 1]$ and $[2, 3]$, since they are both connected, and not properly contained in any other connected subspace of X. □

Next, we deal with the path-connected version.

Definition 3.3.5. Let X be a topological space. For $x, y \in X$, define $x \sim y$ if there is a path from x to y. We will show that $\sim$ is an equivalence relation soon, but assuming that it is, The equivalence classes of $\sim$ are called the **path components** of X. □

Lemma 3.3.6. *$\sim$ is an equivalence relation.*

Proof. For all $x \in X$, there is a constant path $\gamma \colon [0, 1] \to X$ given by $\gamma(t) = x$ for all t. It is clearly continuous.

If there is a path $\gamma \colon [0, 1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$, then define $\gamma' \colon [0, 1] \to X$ by $\gamma'(t) = \gamma(1 - t)$. This is a path from y to x.

Finally, for the transitive property, we can concatenate paths, which was discussed in the last section. □

Proposition 3.3.7. *(1) If A is a path-connected subspace of X, then A is contained in some path component.*
(2) Each path component is a path-connected subspace of X.
(3) Any distinct pair of path components don't intersect, and their union is all of X.

The proof is similar to the previous version, so we we'll leave it as an exercise. Also, similar comments apply here: the path components are exactly the subspaces which are maximal path-connected subspaces.

Corollary 3.3.8. *Every path component of X is contained in a connected component of X.*

Proof. A path component is path-connected, and hence connected, and we've already seen that each connected subspace is contained in some connected component. □

Example 3.3.9. Consider the topologist's sine curve $\overline{X}$ (use the same notation from Example 3.2.5). Then $\overline{X}$ is connected, so it only has one component. The path components of $\overline{X}$ are $X = \overline{X} \cap \{(x, y) \mid x > 0\}$ and the line segment between $(0, -1)$ and $(0, 1)$. Both are path-connected, and the argument in the example shows that there is no path that joins points between these two subsets. □

Now we address the following question: if X is connected, what further property can we require so that X must also be path-connected? There are local versions of (path-)connected that help us here.

Definition 3.3.10. Let X be a topological space and pick $x \in X$.

Then X is **locally connected at** x if, for every neighborhood U of x, there exists a connected neighborhood V of x such that $V \subseteq U$.

Similarly, X is **locally path-connected at** x if, for every neighborhood U of x, there exists a path-connected neighborhood V of x such that $V \subseteq U$.

If X is locally connected at all points, then X is called **locally connected**. Similarly, if X is locally path-connected at all points, then X is called **locally path-connected**. □

We call these "local" properties because they concern the behavior near points, i.e., one may discard parts of the space without affecting the validity of the definition of being locally (path-)connected at x. The original properties, by contrast, are "global" properties.

Proposition 3.3.11. *If X is locally connected, then so is every open subspace. Similarly, if X is locally path-connected, then so is every open subspace.*

Proof. This follows from the definitions: if A is an open subspace and $x \in A$, then a neighborhood U of x in A is also a neighborhood of x in X, so we can find a neighborhood V of x contained in U which is either connected or path-connected (depending on which case we're discussing). □

Now we have 4 properties: connected, path-connected, locally connected, and locally path-connected. We already know that path-connected implies connected, and so locally path-connected implies locally connected.

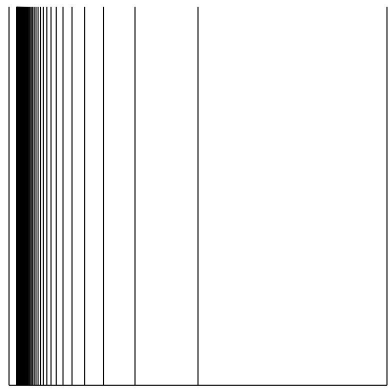
It's a little unintuitive, but the local and global properties are essentially independent of each other.

Example 3.3.12. Let $X = [0, 1] \cup [2, 3]$. Then X is locally path-connected (and hence locally connected) but not path-connected, or even connected. So neither local property implies either global property. □

Example 3.3.13. Let K be the set $\{0\} \cup \{1/n \mid n \in \mathbf{Z}_{>0}\}$. Define the **comb space** X as the following subspace of $\mathbf{R}^2$:

$$X = \{(x, y) \mid x \in K,\ 0 \leq y \leq 1\} \cup \{(x, 0) \mid 0 \leq x \leq 1\}.$$

Here is a visualization:



(I only used $\{0, 1, 1/2, \ldots, 1/50\}$ from K since there isn't enough resolution to distinguish any further, but there would not be a "gap" between the black bar and the rightmost line if we used all of K.)

Then X is path-connected, and hence connected: any point has a path to some point on the bottom segment just by going straight down, and any two points on the bottom segment are connected by a path as well.

However, I claim that X is not locally connected at $a = (0, 1)$ (and hence also not locally path-connected at $(0, 1)$). Let $U = X \cap B(a, 1)$, which is a neighborhood that does not intersect the bottom segment. Now let V be any neighborhood of x that is contained in U. Since V is open, there exists $\epsilon > 0$ such that $X \cap B(a, \epsilon) \subseteq V$. Let n be a positive integer such that $1/n < \epsilon$, and let δ be any irrational number in the interval $(0, 1/n)$. Then $V \cap \{(x, y) \mid x < \delta\}$ and $V \cap \{(x, y) \mid x > \delta\}$ are both open subsets of V, both are nonempty (the first contains $(0, 1)$ and the second contains $(1/n, 1)$), and their union is all of V because no point of V has an irrational x-coordinate (remember, V doesn't intersect the bottom segment). So V is disconnected, and we've established that X is not locally connected at $(0, 1)$. □

We already know that connected does not imply path-connected, so all that remains is asking if locally connected implies locally path-connected. Again there is a counterexample, but we'll skip that discussion.

Now let's see what these local conditions imply.

Proposition 3.3.14. *Let X be a topological space.*

(1) X is locally connected if and only if for every open subspace $U \subseteq X$, every connected component of U is open in X.
(2) X is locally path-connected if and only if for every open subspace $U \subseteq X$, every path component of U is open in X.

Proof. (1) First suppose that X is locally connected. Let U be an open subspace and let C be a connected component of U and pick $x \in C$. Since X is locally connected, there is a connected neighborhood V_x of x such that $V_x \subseteq U$. Since V_x is connected and intersects C, we must in fact have $V_x \subseteq C$. But then $C = \bigcup_{x \in C} V_x$, so C is open in X since each V_x is open in X.

Now suppose that every connected component of every open subspace is open in X. We will show that X is locally connected. Pick $x \in X$ and a neighborhood U of x. Let V be the connected component of U that contains x. By assumption, V is open in X, and hence we see that the definition of locally connected is satisfied.

(2) Same as above, just replace "connected" with "path-connected" and "connected component" with "path component" everywhere. □

Proposition 3.3.15. *Let X be locally path-connected. Then every connected component of X is path-connected. In particular, every path component is a connected component and every connected component is a path component.*

Proof. Let C be a connected component of X and pick $x \in C$. Let P be the path component that contains x. We know that $P \subseteq C$, and I claim that $P = C$. If this fails, then we can write $C \setminus P$ as a union of path components. Since locally path-connected implies locally connected, we know from the previous result that C is open in X, and that each path component of C (these are also path components of X) is open in X, and so we see that P and $C \setminus P$ give a separator for C, which contradicts that C is connected. □

Corollary 3.3.16. *If X is locally path-connected and $U \subseteq X$ is an open connected subspace, then U is path-connected.*

Proof. We've already seen that U is locally path-connected, so we can apply the previous result to U, noting that U is a connected component of itself. □

Remark 3.3.17. Euclidean space $\mathbf{R}^n$ is locally path-connected since we've already seen that (open) balls are path-connected and every neighborhood always contains an open ball.

(1) This previous result then tells us that there is no distinction between connected and path-connected for open subsets of $\mathbf{R}^n$. This is good because path-connected is a more intuitive notion and is probably what we think of when we think about being "connected" in a colloquial sense. Non-open subspaces are a different story (as we've seen with the topologist's sine curve).

(2) "n-Manifolds" are spaces such that every point has a neighborhood that is homeomorphic to $\mathbf{R}^n$ (there are more conditions but we're not defining manifold here). This is a large class of important topological spaces, and the previous result says that they are connected if and only if they are path-connected. Examples of manifolds that we've seen include Euclidean space itself, spheres, and projective space. □

4. Compact spaces

4.1. Definitions and first steps.

The definition of compact was likely covered in your analysis class, but we'll revisit it from the topological perspective.

Recall that an open covering of a topological space X is a collection of open sets $\{U_i\}_{i \in I}$, where I is some index set, such that $\bigcup_{i \in I} U_i = X$.

Definition 4.1.1. Let X be a topological space. We say that X is **compact** if, given any open covering $\{U_i\}_{i \in I}$ of X, there exists a finite subset $J \subseteq I$ such that $\{U_j \mid j \in J\}$ is also an open covering of X. We will call $\{U_j \mid j \in J\}$ a **finite subcovering**. □

Without the symbols: a space is compact if every open covering has a finite subcollection which is also an open covering.

As with our other definitions, being compact is a topological property.

Example 4.1.2. We can't do much with just the definition, but here are some examples of compact spaces.

(1) If X is finite, then it is automatically compact.
(2) If $x_1, x_2, \dots$ is any sequence of points in a space X with a limit point x, then the subspace $\{x, x_1, x_2, \dots\}$ is compact: given an open covering, one of the open sets in that covering contains x and hence contains all but finitely many of the x_i. Now pick one open set in the covering for each of the remaining points to get a finite subcovering. For a concrete example, take $\{0, 1, 1/2, 1/3, 1/4, \dots\}$ as a subspace of $\mathbf{R}$. □

Example 4.1.3. And here are some examples of non-compact spaces.

(1) $\mathbf{R}$ is not compact: we can cover it with the open subsets $(i-1, i+1)$ where i varies over all integers. But there's no way to reduce that to a finite subset which also covers $\mathbf{R}$. More generally, $\mathbf{R}^n$ is not compact for any $n \geq 1$ by considering the collection of open balls $\{B(x,1) \mid x \in \mathbf{Z}^n\}$.
(2) Any open interval in $\mathbf{R}$, such as $(0,1)$ is not compact: consider the open covering $\{(1/n, 1) \mid n \in \mathbf{Z}_{>0}\}$. □

Our first main goal is to give necessary and sufficient conditions for subspaces of $\mathbf{R}^n$ to be compact (the Heine–Borel theorem). This will give a large class of examples; in the process, we'll develop some general results. The necessary conditions are easier and apply to all metrizable spaces, so we'll do that first. First, it's useful to extend the covering definition to subspaces of X.

Definition 4.1.4. Let Y be a subspace of X. Let I be an index set and A_i a subset of X for each $i \in I$. We say that $\{A_i\}_{i \in I}$ **covers** Y if $Y \subseteq \bigcup_{i \in I} A_i$. □

Lemma 4.1.5. *Let $Y \subseteq X$ be a subspace. Then Y is compact if and only if for every covering $\{A_i\}_{i\in I}$ of Y such that A_i is an open subset of X for all $i \in I$, there is a finite subset $J \subseteq I$ such that $\{A_j\}_{j \in J}$ covers Y.*

Proof. First suppose that Y is compact. Let $\{A_i\}_{i \in I}$ be a covering of Y where the A_i are open subsets of X. Then $\{A_i \cap Y\}_{i \in I}$ is an open covering of Y, so there is a finite subset $J \subseteq I$ such that $\{A_j \cap Y\}_{j \in J}$ is also an open covering of Y. But then $\{A_j\}_{j \in J}$ covers Y.

Conversely, suppose that every covering of Y by open subsets of X has a finite subset that also covers Y. Let $\{U_i\}_{i \in I}$ be an open covering of Y. By definition of subspace topology, for each $i \in I$, there is an open subset V_i of X such that $U_i = V_i \cap Y$. But then $\{V_i\}_{i \in I}$ covers Y, so by assumption there is a finite subset $J \subseteq I$ such that $\{V_j\}_{j \in J}$ also covers Y. Then $\{U_j\}_{j \in J}$ is also an open covering of Y. □

Proposition 4.1.6. *Let X be a Hausdorff and let $A \subseteq X$ be a compact subspace.*

(1) Given $x \in X \setminus A$, there exist open subsets U, V of X such that $x \in U$, $A \subseteq V$, and $U \cap V = \varnothing$.
(2) A is a closed subset.

Proof. We may assume $A \neq \varnothing$ since the case $A = \varnothing$ is trivial to deal with.

(1) For each $y \in A$, since X is Hausdorff, we can separate x and y by open neighborhoods U_y and V_y, i.e., $x \in U_y$, $y \in V_y$, and $U_y \cap V_y = \varnothing$. Then $\{V_y\}_{y \in A}$ covers A, and by Lemma 4.1.5, there is a finite subset $J \subseteq A$ such that $\{V_y\}_{y \in J}$ also covers A. Now define

$$V = \bigcup_{y \in J} V_y, \qquad U = \bigcap_{y \in J} U_y.$$

Both are open because J is finite, $A \subseteq V$ by definition of J, $x \in U$ because $x \in U_y$ for all y, and finally $U \cap V = \varnothing$ because if $z \in V$, then $z \in V_j$ for some $j \in J$, which means that $z \notin U_j$ and hence $z \notin U$.

(2) For each $x \in X \setminus A$, let b_x be the subset U guaranteed by (1). Then $b_x \cap A = \varnothing$, i.e., $b_x \subseteq X \setminus A$, and $x \in b_x$, so we conclude that $X \setminus A = \bigcup_{x \in X \setminus A} b_x$ is a union of open sets, and hence is open. □

Definition 4.1.7. Let (X, d) be a metric space. A subset $A \subseteq X$ is **bounded (with respect to d)** if there exists a constant N such that $d(x, y) \leq N$ for all $x, y \in A$. □

Warning 4.1.8. Being bounded is not a topological property and heavily depends on the choice of metric! A consequence of HW3, Problem 7 is that for any metric space (X, d), there exists another metric d' such that X is bounded with respect to d', while (X, d') and (X, d) have the same metric topology. On the other hand, there are plenty of metric spaces which are unbounded (for instance, $X = \mathbf{R}$ with the Euclidean metric).

As a concrete example, consider $X = \mathbf{Z}$ as a subset of $\mathbf{R}$. Then it inherits the Euclidean metric d and this gives the discrete topology on X, and X is unbounded with respect to d. But the trivial metric $d'(x, y) = 1$ for all $x \neq y$ also gives the discrete topology on X and X is bounded with respect to d'. □

Proposition 4.1.9. *Let X be a metrizable space. If $A \subseteq X$ is a compact subspace, then A is closed and bounded with respect to every metric that realizes the topology on X.*

Proof. We've already seen that metrizable spaces are Hausdorff, so the previous result implies that A is closed. Now let d be any metric whose metric topology is the one on X. Pick any point $x \in X$ and consider the collection of open balls $B_d(x, n)$ as n varies over all positive integers. This is an open covering of X, so in particular it covers A. Since A is compact, some finite subcollection also covers A; let N be the largest n so that $B_d(x, N)$ is in that finite collection. In particular, $d(x, y) < N$ for all $y \in A$. So if $y, z \in A$, then we have

$$d(y, z) \leq d(x, y) + d(x, z) < 2N$$

so A is bounded by $2N$. □

Now we work towards giving sufficient conditions for compactness, which will take more effort.

Proposition 4.1.10. *If X is compact and $A \subseteq X$ is a closed subspace, then A is also compact.*

Proof. Let $\{U_i\}_{i \in I}$ be a covering of A by open subsets of X. Then $\{X \setminus A\} \cup \{U_i\}_{i \in I}$ is an open covering of X. Since X is compact, there is a finite subcollection that is also an open covering of X. If this collection uses $X \setminus A$, then delete it, otherwise do nothing. The result is a finite subcollection of our original covering that also covers A, so A is compact by Lemma 4.1.5. □

Lemma 4.1.11. *Let B be a basis for the topology on X. Suppose that for every open covering $\{b_i\}_{i \in I}$ of X such that $b_i \in B$ for all $i \in I$, there is a finite subset $J \subseteq I$ such that $\{b_j\}_{j \in J}$ is an open covering for X. Then X is compact.*

In other words, in the definition of compactness, we can always assume that the open sets in the covering come from a particular basis.

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of X. If $U_i = X$ for some i, then $\{X\}$ is a finite subcovering, so we may assume $U_i \neq X$ for all $i \in I$. For each $x \in X$, there exists $i(x) \in I$ such that $x \in U_{i(x)}$, and by Proposition 1.3.6, there exists $b_x \in B$ such that $x \in b_x$ and $b_x \subseteq U_{i(x)}$. Then $\{b_x\}_{x \in X}$ is an open covering of X by basis elements, so by assumption, there is a finite subset $J \subseteq X$ so that $\{b_j\}_{j \in J}$ is also an open covering. But then $\{U_{i(j)}\}_{j \in J}$ is a finite subcovering of our original open covering, so X is compact. □

Proposition 4.1.12. *If $X_1, \dots, X_n$ are compact spaces, then $X_1 \times \cdots \times X_n$ is also compact.*

Proof. By induction, it suffices to prove this in the special case where $n = 2$.

Consider an open covering of $X_1 \times X_2$ of the form $\{U_i \times V_i\}_{i \in I}$ where $U_i \subseteq X_1$ and $V_i \subseteq X_2$ are open for all $i \in I$. By Lemma 4.1.11, it suffices to check that these special kinds of open coverings have finite subcoverings.

Pick $y \in X_2$. Then $X_1 \times \{y\} \cong X_1$ is compact, so there is a finite subset $J(y) \subseteq I$ such that $\{U_j \times V_j\}_{j \in J(y)}$ covers $X_1 \times \{y\}$. If $y \notin V_j$ for some $j \in J(y)$, we can safely omit it and still have a covering of $X_1 \times \{y\}$, so without loss of generality, we can assume that $y \in V_j$ for all $j \in J(y)$. Now define

$$V(y) = \bigcap_{j \in J(y)} V_j.$$

Since $J(y)$ is finite, this is an open set in X_2 which contains y.

Claim: $X_1 \times V(y) \subseteq \bigcup_{j \in J(y)} (U_j \times V_j)$.

To prove this, pick $(x, z) \in X_1 \times V(y)$. Since $X_1 \times \{y\} \subseteq \bigcup_{j \in J(y)} (U_j \times V_j)$, there exists $j \in J(y)$ such that $x \in U_j$. But also $z \in V_j$ since z is in all of the V_j, so $(x, z) \in U_j \times V_j$, which proves the claim.

Next, $\{V(y)\}_{y \in X_2}$ is an open covering of X_2. Since X_2 is compact, there is a finite subset $\{z_1, \dots, z_n\} \subseteq X_2$ so that $\{V(z_1), \dots, V(z_n)\}$ covers X_2. In particular,

$$X_1 \times X_2 = (X_1 \times V(z_1)) \cup \cdots \cup (X_1 \times V(z_n)) \subseteq \bigcup_{j \in J(z_1) \cup \cdots \cup J(z_n)} (U_j \times V_j),$$

so we have found a finite subcovering of $\{U_i \times V_i\}_{i \in I}$. □

Remark 4.1.13. We can go one step further: there is a generalization of Lemma 4.1.11 where B is only assumed to be a subbasis. This generalization is the "Alexander subbasis theorem" which relies on Zorn's lemma. Using that, one can prove Tychonoff's theorem: all products (not just finite ones) of compact spaces are compact.

Zorn's lemma is an important general set theory tool that has plenty of uses in math, so is worth learning. I'll outline the steps of this discussion as an optional homework problem. □

Theorem 4.1.14. *Given real numbers $a \leq b$, the closed interval $[a, b]$ is compact.*

Proof. If $a = b$, this is obvious, so we can assume $a < b$.

Let $\{U_i\}_{i \in I}$ be an open covering of $[a, b]$. If $a \leq x < b$, define C_x to be the set of y such that $x < y \leq b$ and $[x, y]$ can be covered by finitely many of the U_i (i.e., there exists a finite subset $J \subseteq I$ such that $[x, y] \subseteq \bigcup_{j \in J} U_j$).

Claim 1: For all $x \in [a, b)$, C_x is not empty.

Proof of Claim 1: There exists $i \in I$ such that $x \in U_i$. In particular, since $x \neq b$, and U_i is open, there exists $\epsilon > 0$ such that $[x, x + \epsilon) \subseteq U_i$. Then $x + \epsilon/2 \in C_x$ because $\{U_i\}$ covers $[x, x + \epsilon/2]$. QED

In particular, $C_a \neq \varnothing$, so we can define $c = \sup C_a$.

Claim 2: $c \in C_a$.

Proof of Claim 2: There exists $i \in I$ such that $c \in U_i$. Since $c > a$ by definition, there exists $\epsilon > 0$ such that $(c-\epsilon, c] \subseteq U_i$. Now, $C_a \cap (c-\epsilon, c] \neq \varnothing$: if it were empty, then $c-\epsilon$ would also be an upper bound for C_a, contradicting that $c = \sup C_a$. So pick $d \in C_a \cap (c-\epsilon, c]$. By definition, $[a, d]$ can be covered by finitely many open sets $U_{j_1}, \ldots, U_{j_n}$ for some $j_1, \ldots, j_n \in I$. But now $[a, c] = [a, d] \cup [d, c]$ is covered by $U_{j_1}, \ldots, U_{j_n}, U_i$, so $c \in C_a$. QED

Claim 3: $c = b$.

Proof of Claim 3: Suppose that $c \neq b$. Then from Claim 1, we have $C_c \neq \varnothing$, so we can pick $y \in C_c$. By Claim 2, $[a, c]$ can be covered by finitely many of the U_i, and by definition, $[c, y]$ can be covered by finitely many of the U_i. So $[a, y] = [a, c] \cup [c, y]$ can also be covered by finitely many of the U_i, which means that $y \in C_a$. But $y > c$, which contradicts that c is an upper bound of C_a. QED

Finally, putting everything together, we conclude that $b \in C_a$, which by definition means that $[a, b]$ can be covered by finitely many of the U_i. Since we can prove this for any open covering, we conclude that $[a, b]$ is compact. □

Theorem 4.1.15 (Heine–Borel)**.** *Let $A \subseteq \mathbf{R}^n$ be a subset. Then A is compact if and only if A is closed and bounded (with respect to the Euclidean metric d or the square metric ρ).*

To explain the parenthetical remark: for subsets of $\mathbf{R}^n$, being bounded in the Euclidean metric d is equivalent to being bounded in the square metric ρ because of the following identity:

$$\rho(x, y) \leq d(x, y) \leq \sqrt{n}\rho(x, y).$$

Proof. If A is compact, then A is closed and bounded with respect to every metric by Proposition 4.1.9.

Conversely, suppose that A is closed and that there exists a constant N so that $\rho(x, y) \leq N$ for all $x, y \in A$. Pick a point $a \in A$ and let $P = N + \rho(\mathbf{0}, a)$ where $\mathbf{0}$ is the origin. Then for any $x \in A$, we have

$$\rho(\mathbf{0}, x) \leq \rho(\mathbf{0}, a) + \rho(a, x) \leq P,$$

so in particular, if $x = (x_1, \ldots, x_n)$, then $|x_i| \leq P$ for all $i = 1, \ldots, n$, so that $A \subseteq [-P, P]^n$. Next, by Theorem 4.1.14, $[-P, P]$ is a compact subspace of $\mathbf{R}$, so by Proposition 4.1.12, $[-P, P]^n$ is also compact. Furthermore, the product topology on $[-P, P]^n$ is the same as the subspace topology coming from being a subset of $\mathbf{R}^n$ (exercise). By definition of subspace topology, A is also closed in $[-P, P]^n$. Finally, we use Proposition 4.1.10 to conclude that A is compact. □

Remark 4.1.16. We didn't use the Euclidean metric in the previous proof, but we mention it in the statement since it's usually the metric that is used in practice. □

Example 4.1.17. Some examples which are compact using the Heine–Borel theorem:

(1) The n-sphere $\mathrm{S}^n \subset \mathbf{R}^{n+1}$ is compact. To see it is closed, observe that it is the preimage of $\{1\}$ under the continuous function $f\colon \mathbf{R}^{n+1} \to \mathbf{R}$ given by $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$. Bounded is straightforward: the distance between any two points is at most 2 (by the triangle inequality).

(2) The set of all $n\times n$ real matrices can be identified with $\mathbf{R}^{n^2}$ by choosing an order of the entries. The resulting topology and Euclidean metric doesn't depend on the actual order. Recall that a matrix A is **orthogonal** if $AA^T = I_n$ where T means transpose and I_n is the $n\times n$ identity matrix. The **orthogonal group** is denoted $\mathbf{O}_n(\mathbf{R})$, and is the subset of orthogonal matrices. Then $\mathbf{O}_n(\mathbf{R})$ is closed and bounded (why?), and hence compact.

(3) As a variation of that, we can identify the complex numbers $\mathbf{C}$ with $\mathbf{R}^2$ by considering the real and imaginary part as coordinates. So the set of all $n\times n$ complex matrices can be identified with $\mathbf{R}^{2n^2}$. Recall that a complex matrix A is **unitary** if $AA^* = I_n$ where $*$ means to take the transpose of A but also apply complex conjugation to each entry. The **unitary group** is denoted $\mathbf{U}(n)$, and is the subset of unitary matrices. Then $\mathbf{U}(n)$ is also closed and bounded (why?) and hence compact. □

Here's another useful general fact about compactness and its corollaries.

Proposition 4.1.18. *Let $f\colon X\to Y$ be a continuous function. If X is compact, then $f(X)$ is also compact. More generally, if A is a compact subspace of X, then $f(A)$ is compact.*

Proof. Let $\{U_i\}_{i\in I}$ be a covering of $f(X)$ by open subsets of Y. Then $\{f^{-1}(U_i)\}_{i\in I}$ is an open covering of X, and since X is compact, there is a finite subset $J\subseteq I$ so that $\{f^{-1}(U_j)\}_{j\in J}$ covers X. But then $\{U_j\}_{j\in J}$ covers $f(X)$: given $f(x)\in f(X)$, we know that there exists $j\in J$ so that $x\in f^{-1}(U_j)$; then $f(x)\in U_j$.

For the second statement, apply the first statement to the function $f|_A\colon A\to Y$. □

Corollary 4.1.19. *If X is compact and $\sim$ is an equivalence relation, then $X/{\sim}$ is also compact.*

Proof. The quotient map $\pi\colon X\to X/{\sim}$ is continuous and surjective. □

Example 4.1.20. (1) As before, identify the space of $n\times n$ matrices with $\mathbf{R}^{n^2}$. Let $\mathbf{GL}_n(\mathbf{R})$ be the subset of invertible matrices. Then $\mathbf{GL}_n(\mathbf{R})$ is not compact: the determinant is a continuous function $\mathbf{R}^{n^2}\to\mathbf{R}$ because it is a polynomial, and the image of $\mathbf{GL}_n(\mathbf{R})$ is $\mathbf{R}\setminus\{0\}$, which is not compact.

(2) In Example 2.4.10, we discussed several spaces (Möbius band, torus, Klein bottle) which are quotients of $[0,1]\times[0,1]$. Since the latter is compact, so are these quotient spaces.

(3) Recall that real projective space $\mathbf{RP}^n$ is the quotient of $\mathbf{R}^{n+1}\setminus\{\mathbf{0}\}$ by the equivalence relation $x\sim y$ if x is a scalar multiple of y. While $\mathbf{R}^{n+1}\setminus\{\mathbf{0}\}$ is not compact, the subspace S^n is. Furthermore, every equivalence class has a representative in S^n (in fact, two of them) because $x\sim x/|x|$, so $\mathbf{RP}^n$ is the image of S^n under the quotient map. We conclude that $\mathbf{RP}^n$ is compact.

The next result tells us that if we restrict $\sim$ to S^n, then we also have a homeomorphism $\mathrm{S}^n/{\sim}\to\mathbf{RP}^n$.

(4) We can also define a complex variant of projective space as follows. As before, we identify $\mathbf{C}$ with $\mathbf{R}^2$ to give it a topology. Then put an equivalence relation on $\mathbf{C}^{n+1}\setminus\{\mathbf{0}\}$ by $x\sim y$ if x is a *complex* scalar multiple of y. The quotient is called **complex projective space** and is denoted $\mathbf{CP}^n$. We can modify the argument above to show that $\mathbf{CP}^n$ is also compact.

I'll remark that $\mathbf{CP}^1\cong\mathrm{S}^2$ but that for $n\geq 2$, the spaces $\mathbf{CP}^n$ are new (for us).

(5) Pick nonnegative integers $k \leq n$ and consider $k \times n$ matrices with real entries. As before we can identify this with $\mathbf{R}^{kn}$ to give it a topology. Let X be the subset of matrices of rank k and define an equivalence relation $x \sim y$ if x and y have the same row space. The quotient space $X/\sim$ is the **Grassmannian** $\mathbf{Gr}_k(\mathbf{R}^n)$ and it is in natural bijection with the set of k-dimensional subspaces of $\mathbf{R}^n$. I will leave it as an exercise to prove that $\mathbf{Gr}_k(\mathbf{R}^n)$ is compact. Note that X is *not* compact (why?).

Similarly, if we use matrices with complex entries, we can define a complex variant $\mathbf{Gr}_k(\mathbf{C}^n)$ and it is also compact. When $k = 1$, we have $\mathbf{Gr}_1(\mathbf{R}^{n+1}) = \mathbf{RP}^n$ and $\mathbf{Gr}_1(\mathbf{C}^{n+1}) = \mathbf{CP}^n$ by definition. □

Corollary 4.1.21. *Let $f\colon X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff, then f is a homeomorphism.*

Proof. It suffices to show that if $A \subseteq X$ is closed, then $f(A)$ is closed in Y (we usually use open sets, but this is equivalent since $f(X \setminus A) = Y \setminus f(A)$ because f is a bijection). Since X is compact, A is also compact. But then $f(A)$ is compact by Proposition 4.1.18, and so is closed in Y by Proposition 4.1.6. □

Example 4.1.22. Let's revisit Example 2.4.7. The setup is that $X = [0, 1]$, we define $\sim$ by $0 \sim 1$, and we have a continuous bijection $g\colon X/\sim \to \mathrm{S}^1$ given by $g(x) = (\cos(2\pi x), \sin(2\pi x))$. We've already seen that $[0, 1]$ is compact, and so $[0, 1]/\sim$ is also compact. Finally, S^1 is Hausdorff since it's metrizable (being a subspace of $\mathbf{R}^2$), so the previous result automatically implies that g is a homeomorphism. □

Finally, we generalize a result from calculus.

Theorem 4.1.23 (Extreme value theorem)**.** *Let X be a compact space and let $f\colon X \to \mathbf{R}$ be a continuous function. If $c = \inf f(X)$, then there exists $x \in X$ such that $f(x) = c$. Similarly, if $d = \sup f(X)$, then there exists $x \in X$ such that $f(x) = d$.*

A familiar case is when $X = [a, b]$ is a closed interval.

Proof. We have seen that $f(X)$ is compact. Hence it must be a closed and bounded subspace of $\mathbf{R}$. In particular, bounded implies that $c > -\infty$ and closed implies that c is in the image. Similarly, $d < \infty$ and hence is also in the image. □

We'll end with one last useful property about compact spaces.

Proposition 4.1.24. *Let X be a compact space and let $A_1, A_2, \ldots$ be nonempty closed subsets such that $A_1 \supseteq A_2 \supseteq \cdots$. Then $A = \bigcap_{n=1}^{\infty} A_n$ is non-empty.*

Proof. The subsets $U_i = X \setminus A_i$ are open and $\bigcup_{n=1}^{\infty} U_n = X \setminus A$. If A is empty, then $\{U_n\}$ is an open covering of X, and hence there is a finite subcovering. But $U_1 \subseteq U_2 \subseteq \cdots$, so actually a finite subcovering implies that $X = U_n$ for some n, which implies that A_n is empty. □

4.2. **Variants of compactness.** There are some variations of the notion of compactness which we'll briefly address here.

Recall that given a subset A of a topological space X, x is a limit point of A if $x \in \overline{A \setminus \{x\}}$, or equivalently, every neighborhood of x intersects $A \setminus \{x\}$.

Definition 4.2.1. A topological space X is **limit point compact** if every infinite subset of X has a limit point. □

Proposition 4.2.2. *If X is compact, then X is also limit point compact.*

Proof. Let A be a subset of X with no limit point. We will show that A must be finite. Recall that the closure of a set is obtained by adding all of its limit points (see Proposition 1.6.9), so A must already be closed. Furthermore, if $x \in A$, then there exists a neighborhood U_x of x such that $U_x \cap A = \{x\}$ (because x is not a limit point of A). Hence we get an open covering $\{U_x\}_{x \in A} \cup \{X \setminus A\}$ for X. Since X is compact, it has a finite subcovering, but that is only possible if A is finite (because if $x, y \in A$ and $x \neq y$, then $y \notin U_x$). $\square$

Definition 4.2.3. A topological space X is **sequentially compact** if, given any sequence of points $x_1, x_2, \dots$, there is a subsequence $x_{n_1}, x_{n_2}, \dots$ $(1 \le n_1 < n_2 < \cdots)$ that converges. $\square$

We're going to skip the proof of the following result (it's a bit long, likely was covered in your analysis course, and I don't think we'll make further reference to it...) but state it for completeness.

Proposition 4.2.4. *Let X be a metrizable space. Then the following are equivalent:*

(1) X is compact.
(2) X is limit point compact.
(3) X is sequentially compact.

4.3. Local compactness and the one-point compactification.

Definition 4.3.1. Let X be a topological space and $x \in X$. We say X is **locally compact at** x if there exists a compact subspace A such that A contains a neighborhood of x. We say that X is **locally compact** if it is locally compact at x for all $x \in X$. $\square$

Warning 4.3.2. This is the definition that Munkres gives, but other texts use variations of this definition which are not equivalent. If X is Hausdorff (generally what we will assume in this section) then many other versions are in fact equivalent. $\square$

Proposition 4.3.3. *Every compact space is locally compact.*

Proof. In the definition, take A to be the whole space. $\square$

Example 4.3.4.
(1) $\mathbf{R}^n$ is locally compact: any $x = (x_1, \dots, x_n)$ is contained in the compact subspace $[x_1-1, x_1+1] \times \cdots \times [x_n-1, x_n+1]$ which contains the neighborhood $(x_1-1, x_1+1) \times \cdots \times (x_n-1, x_n+1)$.
(2) $\mathbf{Q}$ is not locally compact at any point x. For instance, suppose $\mathbf{Q}$ is locally compact at x so that there is a neighborhood U of x in $\mathbf{Q}$ contained in a compact subspace $A \subseteq \mathbf{Q}$. Then there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \cap \mathbf{Q} \subseteq U$. Furthermore, since A is also a compact subspace of $\mathbf{R}$, it must be closed in $\mathbf{R}$. But the closure of $(x-\epsilon, x+\epsilon) \cap \mathbf{Q}$ in $\mathbf{R}$ is $[x-\epsilon, x+\epsilon]$ and hence contains irrational numbers, so A cannot be closed in $\mathbf{R}$.
(3) Let $X = \mathbf{R} \times \mathbf{R} \times \cdots = \{(x_1, x_2, \dots) \mid x_i \in \mathbf{R}\}$ be a product of copies of $\mathbf{R}$, one for each positive integer. Then X is not locally compact at any point: every neighborhood contains a basis element $U_1 \times U_2 \times \cdots$ where $U_i = \mathbf{R}$ for all but finitely many i. If A is any subspace that contains this neighborhood and i is an index such that $U_i = \mathbf{R}$, then $\pi_i(A) = \mathbf{R}$ which is not compact, and hence A cannot be compact.

   In particular, an infinite product of locally compact spaces does not have to be locally compact. $\square$

Next, we'll discuss how to embed a space X into a compact space Y.

This is a general theme and there are many ways to do this depending on the goal. First, why would we want to do this? One advantage of compact spaces, as we have seen, is that infinite sets have limit points, and of course, this can fail in non-compact spaces. One important use of topological spaces is when their points represent some kind of objects that we are interested in studying (this is the notion of a *moduli space*); in that case, taking a limit might have some meaning and we would like for them to exist.

Example 4.3.5. We've seen the Grassmannian $\mathbf{Gr}_k(\mathbf{R}^n)$ in Example 4.1.20. This is a topological space such that each point represents a k-dimensional linear subspace of $\mathbf{R}^n$ and is a prototypical example of a moduli space. We've already said that this is a compact space though, so limits exist and there is no need to "fix" it. □

Example 4.3.6. Consider $X = \{(x, y) \in \mathbf{R}^2 \mid x \neq y\}$. By definition, the points of X represent ordered pairs of *distinct* real numbers, or maybe the positions of two particles living in 1 dimension. A sequence of points in X can be interpreted as the positions of these points over time. If they're moving close to each other, for example, we might have the sequence $(-1/n, 1/n) \in X$ for n a positive integer, then there's no limit. However, it might be useful, maybe for some specific calculation, to try to realize X as a subspace of some compact space Y so that there will be a limit (the meaning of this limit will depend on what Y is). □

We might also want to require that this larger space is Hausdorff: as we've seen, in these spaces, limits are unique when they exist. This brings us to the next definition.

Definition 4.3.7. Let X be a topological space which is a subspace of Y. If Y is compact, Hausdorff, and $\overline{X} = Y$, then we call Y a **compactification of** X. If, in addition, $Y \setminus X$ is a single point, then Y is called a **one-point compactification of** X. □

Adding just one point isn't particularly enlightening in regards to our example of moduli spaces since the limits that didn't exist before now do exist, but have the same value. But it's a good starting point and will have some uses, so let's continue.

The next result shows that a one-point compactification is unique (up to homeomorphism) if it exists so we can call it "the" one-point compactification. However, it need not exist.

Proposition 4.3.8. *Suppose that Y and Y' are both one-point compactifications of X. Then there is a homeomorphism $h\colon Y \to Y'$ which is the identity function on X.*

Proof. Let p be the unique point of $Y \setminus X$ and let p' be the unique point of $Y' \setminus X$. Define $h\colon Y \to Y'$ by

$$h(x) = \begin{cases} x & \text{if } x \in X \\ p' & \text{if } x = p \end{cases}.$$

We claim that if $U \subseteq Y$ is open, then $h(U)$ is open in Y'. If $U \subseteq X$, then this holds by definition. Otherwise, $p \in U$. Now set $Z = Y \setminus U$, which is closed in Y. Since Y is compact, so is Z (Proposition 4.1.10). Furthermore, Z is a subset of X because $p \notin Z$, so $h(Z)$ is the same as Z and so is also compact. Since Y' is Hausdorff, that implies that $h(Z)$ is closed in Y' (Proposition 4.1.6). Finally, h is a bijection, so $h(U) = Y' \setminus h(Z)$ is open, and our claim is proven. This shows that h^{-1} is continuous.

Note that the inverse $h^{-1}\colon Y' \to Y$ has the same description as h: identity on X and $h^{-1}(p') = p$. In particular, the above argument with the roles of Y and Y' reversed shows that if $U \subseteq Y'$ is open, then $h^{-1}(U)$ is open in Y. So h is also continuous and we're done. □

Now let's address existence. We start with a general construction.

Definition 4.3.9. Let X be a topological space. The **Alexandroff extension** of X is $X^* = X \amalg \{\infty\}$ (here ∞ just means a new point not belonging to X). A subset $U \subseteq X^*$ is open in the following two cases:

(1) U is a subset of X which is open.
(2) $U = X^* \setminus A$ where $A \subseteq X$ is closed and compact ($A = \varnothing$ is allowed). □

We will informally call these subsets "type 1" and "type 2". We will show soon that this is actually a topology on X^*.

Remark 4.3.10. If X is Hausdorff, then requiring that A be closed in case 2 above is redundant because all compact subsets are closed. But in general we need it. □

Proposition 4.3.11. *The collection of subsets in the previous definition is a topology on X^*. Furthermore, the subspace topology on $X \subseteq X^*$ agrees with the original topology on X.*

Proof. First we check the topology axioms:

(1) $\varnothing$ is open because it is a type 1 open set.
(2) X^* is open because it is type 2.
(3) For finite intersections, we just need to check the intersection of two of these sets is again open. There are 3 cases to consider:
    (a) The intersection of two type 1 open sets is again a type 1 open set since we started with a topology on X.
    (b) Let $U \subseteq X$ be type 1 and $V = X^* \setminus A$ be type 2. Then $U \cap V = U \cap (X \setminus A)$ which is of type 1 because A is closed in X.
    (c) If $U = X^* \setminus A$ and $V = X^* \setminus B$ where A, B are closed and compact subsets of X, then
    $$U \cap V = X^* \setminus (A \cup B)$$
    and the union of two closed and compact subspaces is again closed and compact, so it is again of type 2.
(4) Now consider unions. The union of type 1 sets is again type 1, so we only need to consider unions that involve at least one type 2 set.

    Pick two index sets I and J with $J \neq \varnothing$. Let $\{U_i\}_{i \in I}$ be a collection of type 1 open sets and $\{X^* \setminus A_j\}_{j \in J}$ be a collection of type 2 open sets. Let $U = \bigcup_{i \in I} U_i$ and $A = \bigcap_{j \in J} A_j$. Since each A_j is closed, A is also closed in X. Since A is a closed subset relative to a compact space (any of the A_j), A is also compact. Next, U is open in X, so $A \setminus U = A \cap (X \setminus U)$ is also closed in X and compact. In particular, the union we're interested in is
    $$\bigcup_{i \in I} U_i \cup \bigcup_{j \in J} (X^* \setminus A_j) = U \cup (X^* \setminus A) = X^* \setminus (A \setminus U)$$
    which is of type 2.

Now we consider the second statement. Each open set in X is a type 1 open subset of X^*, so the subspace topology on X refines the original topology. On the other hand, the only potentially new open sets come from type 2, but $X \cap (X^* \setminus A) = X \setminus A$ which is open in the original topology on X because A is closed in X, so we don't get any new open sets, and the two topologies are the same. □

Note that if $X \cong Y$, then $X^* \cong Y^*$ (check this). Here are some further properties of X^*.

Proposition 4.3.12. *(1) X^* is compact.*
(2) If X is locally compact and Hausdorff, then X^* is Hausdorff.

Proof. (1) Pick an open covering of Y. It must have at least one open set of type 2 because the type 1 sets do not contain ∞. Let $X^* \setminus A$ be one of these type 2 open sets in the covering. Then the rest of the open sets cover A. Since A is compact, we can find a finite subcollection that also covers A. By adding back $X^* \setminus A$, we get a finite subcovering of the original open covering of X^*.

(2) Pick two points in X^*. If both points belong to X, then we can separate them using open sets in X since X is Hausdorff. But these are type 1 open sets in X^*, so they can still be separated.

Otherwise our points are x (where $x \in X$) and ∞. Since X is locally compact, there exists a neighborhood U of x and a compact subspace A of X such that $U \subseteq A$ (A is automatically closed because X is Hausdorff). Now x and ∞ are separated in X^* by U and $X^* \setminus A$. □

Corollary 4.3.13. *Let X be locally compact and Hausdorff space which is not compact. Then X^* is the one-point compactification of X.*

Proof. We just need to show that ∞ is a limit point of X. Every neighborhood of ∞ is of the form $X^* \setminus A$ where A is a compact subspace of X. Since X is *not* compact, $X^* \setminus A$ always intersects X. So we're done. □

Example 4.3.14. Take $Y = \mathrm{S}^1$ and take $X = Y \setminus \{(1, 0)\}$. Then Y is a one-point compactification of X. This is not really an interesting statement. But we also know that $X \cong (0, 1)$ because we have the homeomorphism $f : (0, 1) \to X$ given by $f(x) = (\cos(2\pi x), \sin(2\pi x))$. So the circle is a one-point compactification of a bounded open interval, or we can write $\mathrm{S}^1 \cong (0, 1)^*$, and this statement contains much more content. □

Example 4.3.15. Since $(0, 1) \cong \mathbf{R}$, we see that $\mathrm{S}^1 \cong \mathbf{R}^*$. More generally, $\mathrm{S}^n \cong (\mathbf{R}^n)^*$,[3] which follows from an optional problem in HW2: we define a function $f : \mathrm{S}^n \setminus \{(0, 0, \dots, 1)\} \to \mathbf{R}^n$ as follows. Given $a \in \mathrm{S}^n \setminus \{(0, \dots, 1)\}$, draw the line in $\mathbf{R}^{n+1}$ through $(0, \dots, 1)$ and a. It intersects the hyperplane defined by $x_{n+1} = 0$ in exactly one point $(b_1, \dots, b_n, 0)$ and we set $f(a) = (b_1, \dots, b_n)$. Then f is a homeomorphism and S^n is compact and Hausdorff, so it must be the one-point compactification of $\mathbf{R}^n$.

The function f is called **stereographic projection**. □

Example 4.3.16. If $X = (0, 1) \cup (2, 3)$, then X^* is homeomorphic to the union of two circles in the plane which are tangent at one point: deleting this point gives two disjoint copies of something homeomorphic to $\mathrm{S}^1 \setminus \{(1, 0)\}$ which is homeomorphic to each of $(0, 1)$ and $(2, 3)$. More generally, if X is a disjoint union of X_1 and X_2, then X^* is the quotient of $X_1^* \amalg X_2^*$ where the new points in both spaces are made equivalent. □

[3]This is an unfortunate clash of notation: $*$ also denotes the linear dual of a vector space, but we're not going to use that in this course.

Example 4.3.17. It's also good to keep in mind that compactifications in general are not unique. For example, S^1 is the one-point compactification of $(0,1)$, but $[0,1]$ is also a compactification of $(0,1)$ and $[0,1] \ncong \mathrm{S}^1$. $\square$

Remark 4.3.18. The one-point compactification is as small as possible. We can make this precise as follows (details left as an exercise): if Y is any compactification of X and Y' is a one-point compactification of X, then there exists a surjective and continuous function $f\colon Y \to Y'$ which is the identity on X. Time permitting, we'll also study the "Stone–Čech compactification" which is as large as possible (when it exists, it surjects onto any other compactification). $\square$

We end this section with an application of the one-point compactification that allows us to give different formulations of when a space is locally compact and Hausdorff.

Proposition 4.3.19. *Let X be Hausdorff. Then X is locally compact if and only if for all $x \in X$ and all neighborhoods U of x, there exists a neighborhood V of x such that $\overline{V} \subseteq U$ and $\overline{V}$ is compact.*

Proof. Suppose that X is locally compact. Then X^* is Hausdorff and compact by the previous result. Next, set $A = X^* \setminus U$. This is closed in X^* and hence compact. Furthermore, by Proposition 4.1.6(a), we can find open sets V, W such that $x \in V$, $A \subseteq W$, and $V \cap W = \varnothing$. Then $V \subseteq X$ and is a neighborhood of x. Furthermore, since $X^* \setminus W$ is a closed set containing V, we see that $\mathrm{Cl}_{X^*}(V) \subseteq X \setminus W \subseteq U$. Again, $\mathrm{Cl}_{X^*}(V)$ is compact because it is a closed subset of the compact space X^*. Finally, $\overline{V} = \mathrm{Cl}_{X^*}(V) \cap X = \mathrm{Cl}_{X^*}(V)$, so we're done.

On the other hand, the second property implies that X is locally compact at x for any x: taking $U = X$, we then have a compact subspace $\overline{V}$ that contains the neighborhood V of x. $\square$

Corollary 4.3.20. *Let X be a locally compact space.*

(1) If Y is a closed subspace of X, then Y is locally compact.
(2) If Y is an open subspace of X and we also assume that X is Hausdorff, then Y is locally compact.

Proof. (1) Pick $x \in Y$. Since X is locally compact, there exists a compact subspace $A \subseteq X$ that contains a neighborhood U of x in X. Then $Y \cap A$ is closed in A and hence is also compact and it contains $Y \cap U$, which is a neighborhood of x in Y.

(2) We use the previous result. So pick $x \in X$ and let U be a neighborhood of x in Y. Since Y is open, U is also open in X. Then there exists a neighborhood V of x in X such that $\mathrm{Cl}_X(V)$ is compact and $\mathrm{Cl}_X(V) \subseteq U$. But V is also a neighborhood of x in Y, and $\mathrm{Cl}_Y(V) = \mathrm{Cl}_X(V) \cap Y = \mathrm{Cl}_X(V)$, so we're done. $\square$

Corollary 4.3.21. *X is locally compact and Hausdorff if and only if there exists a compact Hausdorff space Y and an open subset $U \subseteq Y$ such that $X \cong U$.*

Proof. If X is locally compact and Hausdorff, we take $Y = X^*$. Then X is an open set inside X^* by definition, and we've already proven that X^* is compact and Hausdorff.

The converse follows from the previous result because Y is compact and hence locally compact. $\square$

5. Countability and separation axioms

The properties in this section will lead us toward metrization theorems (what conditions guarantee a space is metrizable?) and some constructions that are important for the study of manifolds (e.g., in differential geometry).

5.1. Countability axioms.

To fix terminology, a set S is **countably infinite** if there exists a bijection between S and the set of positive integers, and a set is **countable** if it is either finite or countably infinite. A set which is not countable is called **uncountable**. The important thing is that if S is countable, then we can enumerate its elements as $\{s_1, \ldots, s_n\}$ or $\{s_1, s_2, s_3, \ldots\}$. Note that uncountable sets exist; $\mathbf{R}$ is an example.

Some notable reminders:

- $\mathbf{Q}$, the set of rational numbers, is countable,
- a finite product of countable sets is countable, and
- a union of countably many countable sets is also countable.
- the set of *finite* subsets of a countable set is countable (**but**, the set of all subsets of a countably infinite set is not!)

Definition 5.1.1. Let X be a topological space and $x \in X$. We say that X **has a countable basis at** x if there exists a countable collection B_x of neighborhoods of x such that for every neighborhood U of x, there exists $b \in B_x$ such that $b \subseteq U$. If this holds for all x, then we say that X is **first-countable** (we also say X satisfies the **first countability axiom**). □

Definition 5.1.2. If X has a countable basis, then X is called **second-countable** (we also say X satisfies the **second countability axiom**). □

Example 5.1.3. $\mathbf{R}^n$ is second-countable: we take the set of open balls $\{B(x, 1/d) \mid x \in \mathbf{Q}^n,\ d \in \mathbf{Z}_{>0}\}$. Since $\mathbf{Q}^n$ is countable, this is a union of countably many countable sets, so is again countable.

Furthermore, the product $X = \mathbf{R} \times \mathbf{R} \times \cdots$ of copies of $\mathbf{R}$, one for each positive integer, is also second-countable. This is a special case of a result we'll prove soon. □

Proposition 5.1.4. *If X is second-countable, then X is first-countable.*

Proof. If B is a countable basis, then for every $x \in X$, we take $B_x = \{b \in B \mid x \in b\} \cup \{X\}$; a subset of a countable set is again countable. □

Proposition 5.1.5. *(1) If X is first-countable, so is every subspace.*
(2) If X is second-countable, so is every subspace.

Proof. In both cases, we intersect the countable set of basis elements with the subspace. □

Proposition 5.1.6. *Let I be a countable index set.*

(1) If $\{X_i\}_{i \in I}$ is a collection of first-countable spaces, then $\prod_{i \in I} X_i$ is first-countable.
(2) If $\{X_i\}_{i \in I}$ is a collection of second-countable spaces, then $\prod_{i \in I} X_i$ is second-countable.

Proof. (1) This is similar to, but slightly easier than (2), so we'll just prove (2).

(2) Let B_i be a countable basis for X_i. For each finite subset $J \subseteq I$, let C_J be the collection of subsets $\prod_{i \in I} U_i$ such that $U_i \in B_i$ for $i \in J$, and $U_i = X_i$ for $i \notin J$. Then C_J is in bijection with $\prod_{i \in J} B_i$, so is countable. So $\bigcup_J C_J$ is also countable as we vary over all finite subsets J, and it is a basis for $\prod_{i \in I} X_i$. □

5.2. **Separation axioms.** Let X be a topological space. If A, B are disjoint subsets of X, then we say that A and B can be **separated by open sets** if there exist open sets U, V such that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. If one (or both) of these subsets is a singleton $\{x\}$, we'll usually just write x instead.

In this language, Hausdorff just means that any pair of distinct points can be separated by open sets. Now we discuss weaker and stronger versions of this property.

Definition 5.2.1. Let X be a topological space.

(1) X is a T_1**-space** if every singleton $\{x\}$ is a closed subset.
(2) X is **regular** if it is a T_1-space, and any point x and closed subset B such that $x \notin B$ can be separated by open sets.
(3) X is **normal** if it is a T_1-space, and any pair of disjoint closed subsets A and B can be separated by open sets. □

The following implications hold (the first two are by definition):

$$\text{normal} \Rightarrow \text{regular} \Rightarrow \text{Hausdorff} \Rightarrow T_1.$$

Once again, we have a bunch of properties and we can ask if the implications we have not discussed are true in general or if there are counterexamples. I'd rather focus on which *implications are true* rather than focus on counterexamples. But if you're curious, I recommend browsing this database of interesting topological spaces: https://topology.jdabbs.com/ There are no explanations there, but you can find some examples in Munkres that do have justifications.

We can rephrase the last two definitions.

Proposition 5.2.2. *Let X be a T_1-space.*

(1) X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subseteq U$.
(2) X is normal if and only if given a closed subset $A \subseteq X$ and an open set U containing A, there is an open set V containing A such that $\overline{V} \subseteq U$.

Proof. The proofs are going to be very similar, so we'll just do (1).

Suppose that X is regular. Pick $x \in X$ and let U be a neighborhood of x. Then $B = X \setminus U$ is a closed set not containing x. So x and B can be separated by open sets V and W, name them so that $x \in V$ and $B \subseteq W$. Then $\overline{V} \subseteq X \setminus W$, which implies that $\overline{V} \subseteq U$.

Conversely, suppose that the second condition holds. Let $x \in X$ be a point and B a closed subset such that $x \notin B$. Then $U = X \setminus B$ is a neighborhood of x, so by assumption, there exists a neighborhood V of x so that $\overline{V} \subseteq U$. Set $W = X \setminus \overline{V}$. Then $x \in V$ and $B \subseteq W$, both V and W are open, and they don't intersect. Hence X is regular. □

Here are some basic properties; we'll skip the proofs since we won't use the results and the proofs don't contain any significantly new ideas at this point.

Proposition 5.2.3. *(1) If X is regular, then so is every subspace of X.*
(2) If $\{X_i\}_{i \in I}$ is a collection of regular spaces, then $\prod_{i \in I} X_i$ is regular.

Our main focus will be on normal spaces and results about them. One thing to be careful of is that general subspaces of normal spaces need not be normal (again, I don't want to dwell on this, but I mention it because one should make note not to try to use this), though closed subspaces of normal spaces are normal (exercise). First, we prove results which show that many examples we've seen are in fact normal.

Proposition 5.2.4. *If X is metrizable, then X is normal.*

Proof. Pick a metric d that gives the topology on X.

Let A, B be disjoint closed subsets of X. Since $X \setminus A$ is open and contains B, for all $b \in B$, there exists $\epsilon_b > 0$ such that $B_d(b, \epsilon_b) \subseteq X \setminus A$. Similarly, for all $a \in A$, there exists $\epsilon_a > 0$ such that $B_d(a, \epsilon_a) \subseteq X \setminus B$. Now for $a \in A$ and $b \in B$, we have $d(a, b) > \max(\epsilon_a, \epsilon_b)$, and this implies that $B_d(a, \epsilon_a/2) \cap B_d(b, \epsilon_b/2) = \varnothing$: if not, then for any z in the intersection, we would have

$$d(a, b) \leq d(a, z) + d(z, b) < \epsilon_a/2 + \epsilon_b/2 \leq \max(\epsilon_a, \epsilon_b)$$

which is a contradiction. Hence if we define

$$U = \bigcup_{a \in A} B_d(a, \epsilon_a/2), \qquad V = \bigcup_{b \in B} B_d(b, \epsilon_b/2),$$

then $A \subseteq U$, $B \subseteq V$, and $U \cap V = \varnothing$. □

Proposition 5.2.5. *If X is compact and Hausdorff, then X is normal.*

Proof. Let A and B be disjoint closed subsets of X. Since X is compact, both A and B are also compact.

Now we'll use Proposition 4.1.6. For each $x \in B$, there exist open sets U_x and V_x such that $x \in U_x$, $A \subseteq V_x$, and $U_x \cap V_x = \varnothing$. Then $\{U_x\}_{x \in B}$ covers B and since B is compact, there is a finite subset $J \subseteq B$ such that $\{U_j\}_{j \in J}$ also covers B. Now define

$$U = \bigcup_{j \in J} U_j, \qquad V = \bigcap_{j \in J} V_j.$$

Then both are open sets, $B \subseteq U$, $A \subseteq V$, and $U \cap V = \varnothing$: if $z \in U$, then $z \in U_j$ for some j, but then $z \notin V_j$ and so $z \notin V$. □

Proposition 5.2.6. *If X is regular and second-countable, then X is normal.*

Proof. Let B be a countable basis for X. Let Y and Z be disjoint closed subsets of X. We want to show that Y and Z can be separated by open sets. We may assume both are nonempty. Since $X \setminus Z$ is open and contains Y, for each $y \in Y$, there is an open neighborhood U of y contained in $X \setminus Z$. By Proposition 5.2.2, since X is regular, there exists a neighborhood V of y such that $\overline{V} \subseteq U$. Then there exists $b_y \in B$ such that $y \in b_y$ and $b_y \subseteq V$.

Since B is countable, the subset $B_Y = \{b_y \mid y \in Y\}$ is also countable (Y need not be countable, but there can be a lot of redundancy). In particular, we can enumerate its elements $B_Y = \{U_1, U_2, U_3, \dots\}$. We have shown above that $\overline{U_n} \cap Z = \varnothing$ for all n and B_Y covers Y.

By a symmetric argument, there is a countable set of basis elements $B_Z = \{V_1, V_2, \dots\}$ such that $\overline{V_n} \cap Y = \varnothing$ for all n and B_Z covers Z.

Next, define

$$U'_n = U_n \setminus (\overline{V_1} \cup \cdots \cup \overline{V_n}), \qquad V'_n = V_n \setminus (\overline{U_1} \cup \cdots \cup \overline{U_n}).$$

Alternatively, $U'_n = U_n \cap (X \setminus (\overline{V_1} \cup \cdots \cup \overline{V_n}))$, which shows that U'_n is open. Similarly, V'_n is also open.

Claim: $U'_n \cap V'_m = \varnothing$ for all n and m.

If not, pick $z \in U'_n \cap V'_m$. First suppose that $n \le m$. Since $z \in U'_n$, we must have $z \in U_n$. On the other hand, since $z \in V'_m$ and $m \ge n$, we must have $z \notin \overline{U_n}$. But $U_n \subseteq \overline{U_n}$, which is a contradiction. If instead $n \ge m$, we can argue similarly, and the claim is proven.

Finally, the claim implies that the following two open sets are disjoint:

$$U = \bigcup_{n \in \mathbf{Z}_{>0}} U'_n, \qquad V = \bigcup_{n \in \mathbf{Z}_{>0}} V'_n.$$

Furthermore, $Y \subseteq U$: if $y \in Y$, then there exists n such that $y \in U_n$, and $y \notin \overline{V_i}$ for all i because $\overline{V_i} \cap Y = \varnothing$, and hence $y \in U'_n$. Similarly, $Z \subseteq V$. □

5.3. **Urysohn's lemma.** Here's a key result about normal spaces which turns out to be rather important in applications despite first appearances.

We're going to make repeated use of Proposition 5.2.2: if X is normal and $A \subseteq U$ where A is closed and U is open, then we can find an open set V such that

$$A \subseteq V \subseteq \overline{V} \subseteq U.$$

Furthermore, a **dyadic rational number** is a rational number whose denominator (in reduced form) is a power of 2. The key in the next proof is that the dyadic rational numbers are still dense in the real numbers.

Theorem 5.3.1 (Urysohn's lemma)**.** *Let X be a T_1-space.*

X is normal if and only if given any pair of disjoint closed sets A, B, there exists a continuous function $f \colon X \to [0,1]$ such that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$.

Proof. Suppose that X is normal.

Step 1: Our first goal is to show, that for each dyadic rational number $r \in (0,1)$, we can find an open set U_r such that:

(I) $\overline{U_r} \subseteq U_s$ if $r < s$, and
(II) $A \subseteq U_r \subseteq X \setminus B$ for all r.

We construct this by induction on the power of 2 in the denominator of r when written in reduced form. First, since A is closed and $X \setminus B$ is open, Proposition 5.2.2 guarantees an open set $U_{1/2}$ such that

$$A \subseteq U_{1/2} \subseteq \overline{U_{1/2}} \subseteq X \setminus B.$$

For the next step, we first apply Proposition 5.2.2 to $A \subseteq U_{1/2}$ to get an open set $U_{1/4}$ such that

$$A \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2}.$$

Next we apply it again to $\overline{U_{1/2}} \subseteq X \setminus B$ to get an open set $U_{3/4}$ such that

$$\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq X \setminus B.$$

I hope the pattern becomes clear. If not, here is the general formal statement.

By induction, let's assume $U_{p/2^d}$ has been constructed for all $1 \le p \le 2^d - 1$ satisfying the conditions above. Now consider $r = q/2^{d+1}$ where q is odd and $1 \le q \le 2^{d+1} - 1$. To avoid cases, we will make the convention that $\overline{U_0} = A$ (even though U_0 won't be defined) and $U_1 = X \setminus B$. For $1 \le q \le 2^{d+1} - 1$, we apply Proposition 5.2.2 to $\overline{U_{(q-1)/2^{d+1}}} \subseteq U_{(q+1)/2^{d+1}}$, to get an open set $U_{q/2^{d+1}}$ such that $\overline{U_{(q-1)/2^d}} \subseteq U_{q/2^{d+1}} \subseteq \overline{U_{q/2^{d+1}}} \subseteq U_{(q+1)/2^d}$.

Step 2: Given sets U_r satisfying (I) and (II), we define $f\colon X \to [0,1]$ as follows. If $x \in U_r$ for all dyadic rationals r, we set $f(x) = 0$. Otherwise, define

$$f(x) = \sup\{t \mid x \notin U_t\}.$$

By (II), we see that $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. We just have to show that f is actually continuous.

Pick $x \in X$. We will show that f is continuous at x. There are 3 cases to consider depending on whether $f(x) = 0$, $f(x) \in (0,1)$, or $f(x) = 1$. Actually, we'll just explain the second case since the other two can be done in a similar way with slightly less words.

So assume that $0 < f(x) < 1$ and pick $\epsilon > 0$. We can find dyadic rational numbers r, s such that

$$f(x) - \epsilon < r < f(x) < s < f(x) + \epsilon$$

Then $x \in U_s$. By definition of f, there is a dyadic rational $t \in (r, f(x))$ such that $x \notin U_t$ (if not, then $\{t \mid x \notin U_t\}$ has a smaller upper bound). By condition (I), $\overline{U_r} \subseteq U_t$, and so $x \notin \overline{U_r}$.

Now define $V = U_s \setminus \overline{U_r}$. This is an open neighborhood of x. If $y \in U_s$, then by (I), $y \in U_t$ for all $t > s$, and so $f(y) \le s$. Also if $y \notin \overline{U_r}$, then again by (I), $y \notin U_t$ for all $t < r$, and so $f(y) \ge r$. Hence $f(V) \subseteq [r, s] \subseteq (f(x) - \epsilon, f(x) + \epsilon)$, and so f is continuous at x.

The other two cases are similar and slightly easier, so we'll skip it. In conclusion, f is continuous.

Now suppose that the second condition holds. Let A and B be disjoint closed subsets of X. If f is a function as promised by the second condition, then define $U = f^{-1}([0, 1/2))$ and $V = f^{-1}((1/2, 1])$. These are disjoint open sets that contain A and B, respectively, so X is normal. □

5.4. **Urysohn metrization theorem.** We're going to apply Urysohn's lemma to give conditions on a space to be metrizable. This is not the best possible result since the conditions are stronger than being metrizable. There are better versions in Munkres' book, but they require more work, so we'll settle on this one due to time constraints, and in the process we'll see some interesting general results anyway.

Theorem 5.4.1 (Urysohn metrization theorem)**.** *If X is regular and second-countable, then X is metrizable.*

The reason this is not the best possible result is that not every metrizable space is second-countable (for example, take any uncountable set with the trivial metric $d(x, y) = 1 - \delta_{x,y}$), so we're assuming more than is absolutely necessary. See Theorem 40.3 of Munkres (Nagata–Smirnov metrization theorem) for what condition replaces "second-countable" to get an if and only if statement.

Let's fix some notation. If J is any index set and Y is any space, then $Y^J = \prod_{j \in J} Y$ where we're just taking products of copies of Y, one for each element of J. When J is the set of positive integers, we will follow Munkres' notation and write Y^ω.

The strategy is to show that X can be embedded into $\mathbf{R}^\omega$ and then we will directly show that $\mathbf{R}^\omega$ is metrizable (remember that we only showed that finite products of metrizable spaces are metrizable; the infinite case does not work in general, so this requires a separate argument).

Definition 5.4.2. Let X be a topological space and let $\{f_i\}_{i \in I}$ be a collection of continuous functions $f_i \colon X \to \mathbf{R}$. We say that $\{f_i\}$ satisfies **property (U)** if, for all $x \in X$ and all neighborhoods U of x with $U \neq X$, there exists $i \in I$ such that $f_i(x) > 0$ and $f_i(y) = 0$ for $y \in X \setminus U$. □

The property is used in Munkres' book, but with no name. I just made up a name (U for Urysohn) so that we don't have to keep repeating the actual definition. So don't expect to see "property (U)" outside of this class. Here's the point:

Proposition 5.4.3. *Let X be a T_1-space and assume $\{f_i\}_{i \in I}$ satisfies property (U). The continuous function $F \colon X \to \mathbf{R}^I$ given by $f(x) = (f_i(x))_{i \in I}$ is an embedding.*

Proof. First, we claim that F is injective. Pick points $x \neq y$. Since X is T_1, $\{y\}$ is closed, so we can take $U = X \setminus \{y\}$; property (U) tells us there exists $i \in I$ such that $f_i(x) > 0$ and $f_i(y) = 0$. In particular, $F(x) \neq F(y)$.

To finish, we will show that for all open sets $U \subseteq X$, the image $F(U)$ is open in $F(X)$. Pick $x \in U$. By property (U), there exists $n \in I$ such that $f_n(x) > 0$ and $f_n(y) = 0$ for $y \notin U$. Define

$$V_x = \{(z_i)_{i \in I} \in F(X) \mid z_n > 0\}.$$

Then V_x is open in $F(X)$ because $V_x = F(X) \cap \pi_n^{-1}((0, \infty))$ where $\pi_n \colon \mathbf{R}^I \to \mathbf{R}$ is the nth projection map. By definition, $F(x) \in V_x$ and $V_x \subseteq F(U)$. Then $F(U) = \bigcup_{x \in U} V_x$ and hence is open in $F(X)$. □

Proposition 5.4.4. $\mathbf{R}^\omega$ *is metrizable.*

Note that for any countably infinite set J, we have $\mathbf{R}^J \cong \mathbf{R}^\omega$ (a homeomorphism comes from picking an enumeration $J = \{j_1, j_2, \dots\}$ and sending $(x_j)_{j \in J}$ to $(x_{j_1}, x_{j_2}, \dots)$.) On the other hand, if J is uncountable, the space $\mathbf{R}^J$ is *not metrizable* (see Example 2 in §21 of Munkres for a proof).

Proof. We'll use bold letters to denote sequences $\mathbf{x} = (x_1, x_2, \dots)$. Define $D \colon \mathbf{R}^\omega \times \mathbf{R}^\omega \to \mathbf{R}_{\geq 0}$ by

$$D(\mathbf{x}, \mathbf{y}) = \sup_n \left\{ \frac{\min(|x_n - y_n|, 1)}{n} \right\}.$$

We need to check two things: D is a metric, and the corresponding metric topology is the product topology on $\mathbf{R}^\omega$.

The first two axioms for a metric are immediate. Let's check the triangle inequality. Pick sequences $\mathbf{x}, \mathbf{y}, \mathbf{z}$. First, for any n, we have

$$\min(|x_n - z_n|, 1) \leq \min(|x_n - y_n|, 1) + \min(|y_n - z_n|, 1).$$

There are a few cases to consider, but it's straightforward so we'll skip the details. In particular, we have

$$D(\mathbf{x}, \mathbf{z}) \leq \sup_n \left\{ \frac{\min(|x_n - y_n|, 1)}{n} + \frac{\min(|y_n - z_n|, 1)}{n} \right\} \leq D(\mathbf{x}, \mathbf{y}) + D(\mathbf{y}, \mathbf{z}).$$

Now we need to show that the metric topology from D is the same as the product topology, i.e., they refine each other.

Let's start with an open set U in the metric topology. For $\mathbf{x} \in U$, there exists $\epsilon > 0$ so that $B_D(\mathbf{x}, \epsilon) \subseteq U$. Pick a positive integer N such that $1/N < \epsilon$. Define

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times \mathbf{R} \times \mathbf{R} \times \cdots$$

(so the first N components are $(x_i - \epsilon, x_i + \epsilon)$ and the rest are $\mathbf{R}$). This is an open set in the product topology containing $\mathbf{x}$; we claim that $V \subseteq B_D(\mathbf{x}, \epsilon)$ (and hence $V \subseteq U$).

If $\mathbf{y} \in V$, then for $i = 1, \dots, N$, we have $\min(|x_i - y_i|, 1)/i < \epsilon/i$ and for $i > N$, we have $\min(|x_i - y_i|, 1)/i \leq 1/i < 1/N$. Hence

$$D(\mathbf{x}, \mathbf{y}) < \sup\{\epsilon, \epsilon/2, \dots, \epsilon/N, 1/N\} \leq \epsilon,$$

and so $y \in B_D(\mathbf{x}, \epsilon)$ as claimed. Hence U is open in the product topology.

Now let U be an open set in the product topology. If $\mathbf{x} \in U$, there is a basis element $U_1 \times U_2 \times \cdots$ containing $\mathbf{x}$, where U_i is open in $\mathbf{R}$ and there is a positive integer N such that $U_i = \mathbf{R}$ for $i > N$. For each $i = 1, \dots, N$, there exists $\epsilon_i > 0$ such that $(x_i - \epsilon_i, x_i + \epsilon_i) \subseteq U_i$. We may pick ϵ_i so that $\epsilon_i < 1$. Now define $\epsilon = \min(\epsilon_1, \epsilon_2/2, \dots, \epsilon_N/N)$.

We claim that $B_D(\mathbf{x}, \epsilon) \subseteq U_1 \times U_2 \times \cdots$. Pick $\mathbf{y} \in B_D(\mathbf{x}, \epsilon)$. For $i \leq N$, we have $\min(|x_i - y_i|, 1)/i < \epsilon \leq \epsilon_i/i$, and so $\epsilon_i > |x_i - y_i|$ (since $\epsilon_i < 1$) and hence $y_i \in U_i$. For $i > N$, $y_i \in U_i$ because $U_i = \mathbf{R}$. So $\mathbf{y} \in U_1 \times U_2 \times \cdots$ as claimed and we see that U is open in the metric topology. ☐

Now we put it all together.

Proof of Urysohn metrization theorem. Let $\{b_1, b_2, \dots\}$ be a countable basis for X. Let I be the set of pairs (m, n) of positive integers such that $\overline{b_m} \subseteq b_n$. For $(m, n) \in I$, we apply Urysohn's lemma to the disjoint closed sets $\overline{b_m}$ and $X \setminus b_n$ to get a continuous function $g_{m,n} \colon X \to [0, 1]$ that is 1 on $\overline{b_m}$ and 0 on $X \setminus b_n$.

I claim that $\{g_{m,n}\}_{(m,n) \in I}$ satisfies property (U). Pick $x \in X$ and a neighborhood U of x with $U \neq X$. Then there is a basis element b_n such that $x \in b_n$ and $b_n \subseteq U$. Since X is regular, using Proposition 5.2.2, we can find a neighborhood V of x such that $\overline{V} \subseteq b_n$. Now let b_m be any neighborhood of x contained in V. Then $\overline{b_m} \subseteq b_n$ and so $g_{m,n}(x) = 1$ and $g_{m,n}$ is 0 outside of b_n (and hence outside of U).

Finally, Proposition 5.4.3 tells us that the function $F \colon X \to \mathbf{R}^I$ given by $F(x) = (g_{m,n}(x))_{(m,n) \in I}$ is an embedding. If I is finite, then $\mathbf{R}^I \cong \mathbf{R}^{|I|}$ is metrizable. Otherwise, I is countably infinite and then $\mathbf{R}^I \cong \mathbf{R}^\omega$, so again is metrizable by Proposition 5.4.4. In any case, we see that X is homeomorphic to a subspace of a metrizable space, so is itself metrizable. ☐

5.5. **Topological manifolds.**

For the last topic of this course, we'll introduce the concept of a (topological) manifold and prove that every compact manifold can be embedded into $\mathbf{R}^N$ for some $N < \infty$ (much more is true but beyond what we have time for). These are very fundamental objects in mathematics so we'll only scratch the surface, but I hope it gives you some idea of how many of the ideas and results we've been proving can be put together.

Definition 5.5.1. Let $n \geq 0$ be a nonnegative integer.

A (nonempty) space X is **locally Euclidean of dimension** n if there is an open covering $\{U_i\}_{i \in I}$ such that each U_i is homeomorphic to an open subset of $\mathbf{R}^n$.

A **topological** n**-manifold** is a space X such that X is Hausdorff, second-countable, and locally Euclidean of dimension n. ☐

An n-manifold is also said to have **dimension** n or be n**-dimensional**.

Remark 5.5.2. (1) There's another definition you can make of a space: ask that there is an open covering by subspaces that are homeomorphic to $\mathbf{R}^n$. This sounds stricter than locally Euclidean, but it's actually equivalent because for a locally Euclidean space, if $x \in U_i$ and U_i is homeomorphic to an open subset of $\mathbf{R}^n$, then there is a neighborhood of x homeomorphic to $(x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_n - \epsilon, x_n + \epsilon)$, and we've already seen in homework that open intervals are homeomorphic to $\mathbf{R}$. We stick with our definition because it's more flexible for definitions and generalizes better when defining other types of manifolds.

(2) If we can find a countable open covering $\{U_1, U_2, \dots\}$ to satisfy the locally Euclidean condition, then second-countability is automatic: since each U_i has a countable basis and we can simply take the union of these bases to get a basis for X. This will automatically hold for all of our examples below, so we won't explicitly need to comment about second-countability. □

Remark 5.5.3. The choice of the open covering is not part of the definition, we only need it to exist. However, there are other types of manifolds, where not only the choice matters for the definition, but the actual embeddings $\varphi_i \colon U_i \to \mathbf{R}^n$ matter too (these are referred to as **charts**), and are considered as part of the object. A commonly studied object is a *smooth manifold* which is a pair consisting of a topological manifold together with a choice of charts so that for all i, j, the compositions $\varphi_j \circ \varphi_i^{-1} \colon \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ are infinitely differentiable functions. In particular, a smooth manifold is a topological manifold together with some additional structure, not just a topological manifold satisfying an extra property (analogous to how a topology is an extra structure we can put on a set), and it is possible that a given topological manifold may have more than one inequivalent smooth structure. This extra structure allows us to do calculus on X. But this is beyond the scope of what we have time for, so I'll refer you to a course/book on differential geometry. □

Let's see some examples:

Example 5.5.4. (1) $\mathbf{R}^n$ is of course an n-manifold. What is not obvious is that $\mathbf{R}^n$ is not an m-manifold for $m \neq n$, i.e., an open subset of $\mathbf{R}^n$ cannot be homeomorphic to an open subset of $\mathbf{R}^m$ if $m \neq n$. It is true (and called "invariance of domain" and usually is proven using tools from algebraic topology), so the dimension of a manifold is well-defined.

(2) The n-sphere S^n is an n-manifold. We've already discussed stereographic projection in Example 4.3.15 which gives a homeomorphism between $\mathbf{R}^n$ and $U_1 = \mathrm{S}^n \setminus \{(0, \dots, 0, 1)\}$. But actually the choice of the "north pole" was not relevant, and we could have done the same with any point. In particular, if you pick the "south pole" $(0, \dots, 0, -1)$, then we have $\mathbf{R}^n \cong U_2 = \mathrm{S}^n \setminus \{(0, \dots, 0, -1)\}$ and $\mathrm{S}^n = U_1 \cup U_2$.

Note that since S^n is compact, it itself cannot be homeomorphic to any open subset of $\mathbf{R}^n$ because of the Heine–Borel theorem.

(3) The set of invertible $n \times n$ matrices $\mathbf{GL}_n(\mathbf{R})$ is itself an open subset of $\mathbf{R}^{n^2}$: it is the preimage of $\mathbf{R} \setminus \{0\}$ under the determinant function (which is continuous). In particular, it is a n^2-manifold.

(4) Real projective space $\mathbf{RP}^n$ is an n-manifold. The main points are handled in an exercise in HW6: in particular, you are asked to prove that it is Hausdorff and that the subspace U_{n+1} of representatives $[x_1 : \cdots : x_{n+1}]$ where $x_{n+1} \neq 0$ is homeomorphic to $\mathbf{R}^n$. But actually there's nothing special about the last coordinate: define U_i to be the set of representatives such that $x_i \neq 0$. The same argument shows that $U_i \cong \mathbf{R}^n$ but also $\mathbf{RP}^n = U_1 \cup \cdots \cup U_{n+1}$.

(5) With more effort, one can prove that the Grassmannian $\mathbf{Gr}_k(\mathbf{R}^n)$ is a $k(n-k)$-manifold. We won't discuss the details, so you can take it as more challenging exercise.
(6) If X is an m-manifold and Y is an n-manifold, then $X \times Y$ is an $(m+n)$-manifold (exercise). So for example, the torus $\mathrm{S}^1 \times \mathrm{S}^1$ is a 2-manifold. $\square$

Here's a few things we can conclude about topological manifolds:

(1) Since they are locally Euclidean, they are also locally compact: for each $x \in X$, there is a neighborhood U of x that is homeomorphic to an open subset of $\mathbf{R}^n$. In such sets, we can always find a compact subspace that contains a neighborhood of any point (for example, a closed ball of radius 1 and the corresponding open ball).
(2) By definition, X is also Hausdorff, so non-compact manifolds always have a one-point compactification X^* (though X^* doesn't have to be a manifold; this fails for $X = \mathrm{S}^5 \times \mathbf{R}$, see https://math.stackexchange.com/q/2102815/8399).
(3) Similarly, since they are locally Euclidean, we can use HW4 #5 to conclude that they are also locally path-connected since open subsets of $\mathbf{R}^n$ are locally path-connected.
(4) In particular, since X is locally path-connected, the connected components of X are the same thing as path-components, and they are all open subsets of X. Since X is second-countable, this implies that X has countably many connected components.

Remark 5.5.5. An interesting question is to try to classify all n-manifolds for a given n. The previous remark tells us that we should focus on connected manifolds, since everything else can be obtained by taking disjoint unions. For $n = 0$, the only example is a single point. For $n = 1$, there are two examples (up to homeomorphism): $\mathbf{R}$ and S^1. Showing that these are the only examples already requires some ideas beyond the scope of this course. You can see https://en.wikipedia.org/wiki/Classification_of_manifolds for an overview of what goes into studying this question for $n \geq 2$. $\square$

There's a lot of interesting basic stuff we could discuss about manifolds, but that's really the content of other courses (like Math 150). Instead, let's move onto proving that a compact manifold can be embedded into $\mathbf{R}^N$ for some $N < \infty$.

This relies on the existence of a "partition of unity". We'll only do this in a special case.

Definition 5.5.6. Let X be a space. If $f\colon X \to \mathbf{R}$ is a continuous function, then the **support** of f, denoted $\mathrm{Supp}(f)$, is the closure of $f^{-1}(\mathbf{R} \setminus \{0\})$. $\square$

Definition 5.5.7. Let X be a space with an open covering $\{U_1, \dots, U_n\}$. A collection of continuous functions $\varphi_i\colon X \to [0,1]$ $(i = 1, \dots, n)$ is a **partition of unity subordinate to** $\{U_i\}$ if:

(1) $\mathrm{Supp}(\varphi_i) \subseteq U_i$ for $i = 1, \dots, n$, and
(2) For all $x \in X$, we have $\sum_{i=1}^n \varphi_i(x) = 1$. $\square$

Lemma 5.5.8. *If X is normal and $\{U_1, \dots, U_n\}$ is an open covering, then there exists an open covering $\{V_1, \dots, V_n\}$ such that $\overline{V_i} \subseteq U_i$ for all i.*

Proof. For $k = 0, \dots, n$, and open subsets $V_1, \dots, V_k$, we will call these open subsets "good" if

$$\{V_i \mid 1 \leq i \leq k\} \cup \{U_i \mid k+1 \leq i \leq n\} = \{V_1, \dots, V_k, U_{k+1}, \dots, U_n\}$$

covers X and $\overline{V_i} \subseteq U_i$ for $i \leq k$.

We will find open subsets $V_1, \dots, V_n$ such that $V_1, \dots, V_k$ is good for all k. We will do this by induction on k. If $k = 0$, the statement is vacuous, so the base case is automatic. So now assume that $n \ge k \ge 1$ and that we have found a good collection $V_1, \dots, V_{k-1}$. Define

$$A = X \setminus (V_1 \cup \cdots \cup V_{k-1} \cup U_{k+1} \cup \cdots \cup U_n),$$

which is a closed subset of X contained in U_k because $X = V_1 \cup \cdots \cup V_{k-1} \cup U_k \cup \cdots \cup U_n$. Since X is normal, by Proposition 5.2.2, there exists an open set, call it V_k, such that $A \subseteq V_k$ and $\overline{V_k} \subseteq U_k$. Since $A \subseteq V_k$, we have that $X = V_1 \cup \cdots \cup V_k \cup U_{k+1} \cup \cdots \cup U_n$, so $V_1, \dots, V_k$ is good. Finally, we have a good collection $V_1, \dots, V_n$, which is exactly what we're looking for. $\square$

Proposition 5.5.9. *If X is normal and $\{U_1, \dots, U_n\}$ is an open covering, then there exists a partition of unity subordinate to $\{U_i\}$.*

Proof. Applying the previous lemma twice gives us open coverings $\{V_1, \dots, V_n\}$ and $\{W_1, \dots, W_n\}$ of X such that $\overline{W_i} \subseteq V_i$ and $\overline{V_i} \subseteq U_i$ for all i. Then for each i, we apply Urysohn's lemma to the disjoint closed sets $\overline{W_i}$ and $X \setminus V_i$ to get a continuous function $\psi_i \colon X \to [0,1]$ such that ψ_i is 1 on $\overline{W_i}$ and 0 on $X \setminus V_i$. Then $\mathrm{Supp}(\psi_i) \subseteq \overline{V_i} \subseteq U_i$.

For all $x \in X$, define $\Psi(x) = \sum_{i=1}^n \psi_i(x)$. Since $\psi_i(x) \ge 0$ for all i and there exists j such that $x \in W_j$, we see that $\psi_j(x) = 1$ and hence $\Psi(x) > 0$ for all x. In particular, we can define

$$\varphi_i(x) = \frac{\psi_i(x)}{\Psi(x)}.$$

Then by definition, $\mathrm{Supp}(\varphi_i) = \mathrm{Supp}(\psi_i) \subseteq U_i$ and $\sum_{i=1}^n \varphi_i(x) = 1$ for all x. $\square$

Theorem 5.5.10. *Let X be a compact m-manifold. Then there exists a positive integer N and an embedding $F \colon X \to \mathbf{R}^N$.*

We can be more specific if we do more work: you can take $N = 2m + 1$. See Corollary 50.8 of Munkres for details.

Proof. By definition, X has an open covering $\{U_i\}_{i \in I}$ such that for all $i \in I$, U_i is homeomorphic to an open subset of $\mathbf{R}^m$. Since X is compact, we can find a finite subcovering, call it $\{U_1, \dots, U_n\}$. Using the homeomorphisms, we have embeddings $g_i \colon U_i \to \mathbf{R}^m$ for $i = 1, \dots, n$.

Next, since X is compact and Hausdorff, we also know that X is normal by Proposition 5.2.5. In particular, the previous result gives us a partition of unity $\{\varphi_1, \dots, \varphi_n\}$ subordinate to $\{U_1, \dots, U_n\}$. For $i = 1, \dots, n$, define $h_i \colon X \to \mathbf{R}^m$ by

$$h_i(x) = \begin{cases} \varphi_i(x) \cdot g_i(x) & \text{if } x \in U_i \\ \mathbf{0} & \text{if } x \in X \setminus \mathrm{Supp}(\varphi_i) \end{cases}$$

where as usual $\mathbf{0} = (0, \dots, 0)$ is the origin. We've defined h_i piecewise on two open sets of X; if $x \in U_i \cap (X \setminus \mathrm{Supp}(\varphi_i))$, then by definition $\varphi_i(x) = 0$, so we see that h_i is well-defined. Since both components are continuous, h_i is a continuous function.

Now take $N = n + nm$; we define

$$F \colon X \to \underbrace{\mathbf{R} \times \cdots \times \mathbf{R}}_{n} \times \underbrace{\mathbf{R}^m \times \cdots \times \mathbf{R}^m}_{n} = \mathbf{R}^N$$

$$F(x) = (\varphi_1(x), \dots, \varphi_n(x), h_1(x), \dots, h_n(x)).$$

Then F is continuous because each of its component functions are continuous. I claim that F is injective. Suppose that $F(x) = F(y)$. Since $\sum_{i=1}^{n} \varphi_i(x) = 1$, there exists j such that $\varphi_j(x) \neq 0$. In particular, $\varphi_j(y) \neq 0$ since $\varphi_j(y) = \varphi_j(x)$, and so $x, y \in U_j$. But then $h_j(x) = \varphi_j(x)g_j(x) = \varphi_j(y)g_j(y) = h_j(y)$, and we can divide to conclude that $g_j(x) = g_j(y)$. Since g_j is an embedding, we conclude that $x = y$.

Finally, X is compact and its image $F(X)$ is Hausdorff, and we have just shown that $F \colon X \to F(X)$ is a continuous bijection, so it is a homeomorphism by Corollary 4.1.21. $\square$