Math 200C, Spring 2022
Homework 2
Due: April 15 11:59PM via Gradescope

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material in Section 3 of the notes (lectures 5–6).

(1) Let n be a square-free integer (i.e., every prime divides n at most once). Let $\mathbf{Q}$ be the rational numbers. Define
$$\mathbf{Q}(\sqrt{n}) = \{a + b\sqrt{n} \mid a, b \in \mathbf{Q}\},$$
which is the splitting field of $x^2 - n$ over $\mathbf{Q}$.
  (a) If $b \neq 0$, show that $a + b\sqrt{n}$ satisfies a unique monic degree 2 polynomial with rational coefficients.
  (b) Determine the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{n})$.
  (Hint: the answer will depend on whether or not $n \equiv 1 \pmod 4$)

(2) Let $\mathbf{k}$ be a field and consider the two rings
$$A = \mathbf{k}[x, y]/(y^2 - x^3), \qquad B = \mathbf{k}[x, y]/(y^2 - x^3 - x^2).$$
They are both domains (you don't have to prove this); show that in both cases the normalization is the subring of the field of fractions generated by the ring and y/x.

Hint: Show that adjoining y/x gives a ring which is isomorphic to a polynomial ring over $\mathbf{k}$ in 1 variable.

(3) (a) Let A a ring and $f = t^n + a_1 t^{n-1} + \cdots + a_n$ be any monic polynomial with coefficients in A. Define the **splitting ring** $S_A(f)$ of f to be
$$S_A(f) = A[\xi_1, \ldots, \xi_n]/I$$
where $\xi_1, \ldots, \xi_n$ are variables, and I is generated by the coefficients of
$$(t - \xi_1) \cdots (t - \xi_n) - f(t)$$
thought of as a polynomial in t. Show that the natural map $A \to S_A(f)$ is integral (you don't need to prove it is injective, though that is true).
  (b) Atiyah–Macdonald, Exercise 5.8.ii.

(4) Atiyah–Macdonald, Exercise 5.12

1. Extra problems (don't submit)

(5) Atiyah–Macdonald, Exercise 5.9