

Math 200C, Spring 2022  
Homework 3  
Due: April 26 11:59PM via Gradescope

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material in Section 4 of the notes (lectures 7–10).

- (1) (a) Let  $A$  a ring and consider the set  $T$  of ideals of  $A$  which are not finitely generated. If  $T \neq \emptyset$ , show that any maximal element (with respect to inclusion) of  $T$  is prime. Deduce that if every prime ideal is finitely generated, then  $A$  is noetherian.  
(Hint: if  $I \in T$  is a maximal element but not prime, there exist  $x, y \notin I$  such that  $xy \in I$ . Show that  $(I : x) := \{a \in A \mid ax \in I\}$  is finitely generated and that  $I + (x)$  can be generated by finitely many elements of  $I$  together with  $x$ ; use these to get a finite generating set for  $I$ .)
- (b) Give an example of a non-noetherian ring that satisfies ACC for prime ideals.
- (2) Let  $\mathbf{k}$  be a field and let  $A = \mathbf{k}[x]$  be the polynomial ring in 1 variable. Consider the Laurent polynomial ring  $\mathbf{k}[x, x^{-1}]$  as an  $A$ -module. Let  $M$  be the quotient of  $\mathbf{k}[x, x^{-1}]$  by the  $A$ -submodule generated by  $(x)$ , so that  $M$  has a  $\mathbf{k}$ -basis  $\{1, x^{-1}, x^{-2}, \dots\}$  with the rule  $x \cdot x^{-i} = x^{-i+1}$  if  $i > 0$  and  $x \cdot 1 = 0$ .  
Show that  $M$  is artinian and not finitely generated.
- (3) (a) Let  $\mathbf{k}$  be a field. Construct a subring  $B \subset \mathbf{k}[x, y]$  which is not noetherian.  
(b) Let  $A$  be a noetherian ring and  $B \subset A$  be a subring, and consider  $A$  as a  $B$ -module in the usual way. Suppose there is a homomorphism of  $B$ -modules  $\psi: A \rightarrow B$  such that  $\psi(b) = b$  for all  $b \in B$ . Show that  $B$  is noetherian.  
(c) Let  $A$  be a noetherian ring and let  $G$  be a finite group of automorphisms of  $A$  such that  $|G|$  (i.e., the result of adding  $|G|$  many instances of 1) is invertible in  $A$ . Let  $B = A^G$  be the subring of elements fixed by all automorphisms.  
Show that

$$\psi(a) = \frac{1}{|G|} \sum_{g \in G} g(a)$$

satisfies the hypotheses of (b) and hence that  $A^G$  is noetherian.

- (4) Let  $\mathbf{k}$  be a field and let  $\mathfrak{m}$  be a maximal ideal of the polynomial ring  $\mathbf{k}[x_1, \dots, x_n]$ . Show that  $\mathfrak{m}$  can be generated by  $n$  polynomials  $f_1, \dots, f_n$  such that  $f_i \in \mathbf{k}[x_1, \dots, x_i]$  for all  $i$  (i.e.,  $f_i$  only has terms that use the first  $i$  variables).
- (5) Atiyah–Macdonald, Exercise 6.1

#### 1. EXTRA PROBLEMS (DON'T SUBMIT)

- (6) Atiyah–Macdonald, Exercise 6.4  
(7) A ring  $A$  is **coherent** if every finitely generated ideal  $I$  is a finitely presented  $A$ -module.

- (a) Suppose that  $A$  is coherent. Let  $f: M \rightarrow N$  be a homomorphism of finitely presented  $A$ -modules. Show that  $\ker f$ ,  $\operatorname{coker} f$ ,  $\operatorname{image} f$  are all finitely presented  $A$ -modules.
- (b) Let  $A$  be a noetherian ring. Show that the polynomial ring in infinitely many variables  $A[x_1, x_2, \dots]$  is coherent.