Math 200C, Spring 2022
Homework 4
Due: May 6 11:59PM via Gradescope

Please do not search for solutions. I would rather help you directly (via office hours or Discord) so that I can calibrate explanations in the notes and lecture. You are free to work with other students, but solutions must be written in your own words. Please cite any sources (beyond the course materials) that you use or any people you collaborated with.

This covers the material up through Section 5 of the notes (lectures 11–13).

(1) Let $\mathbf{k}$ be a field and let $I = (x_1, x_2) \cap (x_3, x_4)$ in the graded ring $A = \mathbf{k}[x_1, x_2, x_3, x_4]$ (with $\deg(x_i) = 1$ for all $i$). Compute the Hilbert series and Hilbert polynomial for $A/I$.

(2) Let $A$ be a finitely generated $\mathbf{Z}_{\geq 0}$-graded $\mathbf{k}$-algebra over a field $\mathbf{k}$ such that $A_0 = \mathbf{k}$, and let $M$ be a finitely generated $\mathbf{Z}_{\geq 0}$-graded $A$-module. Let $f_1, \ldots, f_r \in A$ be a sequence of positive degree homogeneous elements and let $d_i = \deg(f_i)$. We call $f_1, \ldots, f_r$ a **regular sequence on** $M$[1] if for all $i = 1, \ldots, r$, $f_i$ is a nonzerodivisor on $M/(f_1, \ldots, f_{i-1})M$ (for $i = 1$, this is just $M$).

If $f_1, \ldots, f_r$ is a regular sequence on $M$, show that

$$\mathrm{H}_{M/(f_1,\ldots,f_r)M}(t) = \mathrm{H}_M(t)(1 - t^{d_1}) \cdots (1 - t^{d_r}).$$

(3) Let $A = \mathbf{k}[x_1, \ldots, x_n]$ be a polynomial ring over an algebraically closed field $\mathbf{k}$ which is graded in the usual way ($\deg(x_i) = 1$; not necessary, but we do it for simplicity of notation). Let $f_1, \ldots, f_n \in A$ be positive degree homogeneous elements. Consider the following three statements:

(a) $A/(f_1, \ldots, f_n)$ is artinian.

(b) The only point $(\alpha_1, \ldots, \alpha_n) \in \mathbf{k}^n$ such that $f_i(\alpha_1, \ldots, \alpha_n) = 0$ for all $i = 1, \ldots, n$ is $(0, \ldots, 0)$.

(c) $f_1, \ldots, f_n$ is a regular sequence.

Show that (a) and (b) are equivalent and that (c) implies both of them.[2]

(4) Let $t, x_1, \ldots, x_n$ be variables. The elementary symmetric polynomials $e_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$ are defined so that the following identity is true:

$$(t + x_1) \cdots (t + x_n) = t^n + \sum_{i=1}^{n} e_i(x_1, \ldots, x_n) t^{n-i}.$$

In other words, $e_i(x_1, \ldots, x_n)$ is the sum of all possible ways to multiply together $i$ different $x$'s. Show that $e_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$ satisfies condition (b) in the previous problem with $A = \mathbf{k}[x_1, \ldots, x_n]$ (and $\mathbf{k}$ is any field).

[1]This definition makes sense in the non-graded setting, but we should also add the requirement (automatic here) that $(f_1, \ldots, f_r)M \neq M$.

[2]In fact, (c) is also equivalent to (a) and (b), but I don't know if that's easy to show with what we've proven in class so far; if you see a way, let me know.