Artinian Rings

Prop. A = artinian ring, every prime ideal is maximal.

Pf. Let $p \subset A$ be prime. Then A/p is an artinian domain.
Pick $x \in A/p$, $x \neq 0$. Consider $(x) \supseteq (x^2) \supseteq (x^3) \supseteq \cdots$
This stabilizes, so $\exists n > 0$ s.t. $(x^n) = (x^{n+1}) \Rightarrow \exists y \in A/p$ s.t.
$x^n = yx^{n+1} \Rightarrow x^n(1 - yx) = 0 \Rightarrow yx = 1. \Rightarrow x$ unit

$\Rightarrow p$ maximal. $\square$

Prop. A = artinian ring, A has only finitely many maximal ideals

Pf. Suppose not, let A be artinian w/ maximal ideals $m_1, m_2, \ldots$

Consider chain $m_1 \supseteq m_1 \cap m_2 \supseteq m_1 \cap m_2 \cap m_3 \supseteq \cdots$

stabilizes $\Rightarrow \exists n > 0$ s.t. $m_1 \cap \ldots \cap m_n = m_1 \cap \ldots \cap m_n \cap m_{n+1}$

$\Rightarrow m_1 \cap \ldots \cap m_n \subseteq m_{n+1} \Rightarrow \exists i$ s.t. $m_i \subseteq m_{n+1}$

$\Rightarrow m_i = m_{n+1}$ $\rightarrow\leftarrow$ $\square$

Prop. If A artinian ring, then A is also noetherian.

Pf. Suppose A artinian, but not noetherian.
Consider set of all ideals which are not finitely generated.
This is nonempty, so $\exists$ minimal element, call it I.

Claim: If $x \in A$, then $xI = 0$ or $xI = I$.

Pf of claim: If $xI \neq I$, then xI is finitely generated,
let K be the kernel of $\begin{array}{l} I \longrightarrow xI \\ y \longmapsto xy \end{array}$. Then we have

$0 \to K \to I \to xI \to 0$. If K finitely generated, then so is I.

Since I is not, K can't be finitely gen. $\Rightarrow K = I \Rightarrow xI = 0$. $\square$

Define $J = \mathrm{Ann}_A(I) = \{ x \in A \mid xI = 0 \}$.

If $xy \in J$ and $y \notin J$, then

$0 = xyI = x(yI) = xI \Rightarrow x \in J$.

Hence, J is prime ideal.

Hence, A/J is an artinian domain (hence a field) and I is a A/J-module (i.e, vector space).

$\dim_{A/J} I$ is infinite, otherwise I would be finitely gen., but every proper subspace is finite dim. $\Rightarrow\Leftarrow$ Hence A noeth $\square$

<u>Cor.</u> If A artinian ring, then every finitely generated A-module has finite length.

<u>Pf.</u> A is noeth $\Rightarrow$ f.g. A-modules are noeth
A is artinian $\Rightarrow$ f.g A-modules are artinian
$\Bigg\} \Rightarrow$ f.g. modules are finite length $\square$

<u>Thm.</u> A is artinian ring $\Leftrightarrow$ A is noeth & every prime ideal is maximal.

<u>Pf.</u> ($\Rightarrow$) done.

($\Leftarrow$) Suppose A is noeth ring & every prime ideal is maximal.

We can write $0 = I_1 \cap \dots \cap I_r$, I_j are irreducible ideals.

Also, $p_i := \sqrt{I_i}$ is a prime ideal.

$$\Rightarrow \quad \mathfrak{N} := \sqrt{0} = p_1 \cap \dots \cap p_r.$$

Also, $\mathfrak{N}$ = intersection of all prime ideals.

Hence, if p is any prime ideal, $p \cap p_1 \cap \dots \cap p_r = p_1 \cap \dots \cap p_r$

$\Rightarrow p \supseteq p_1 \cap \dots \cap p_r \Rightarrow \exists i$ s.t. $p_i \subseteq p \Rightarrow p_i = p$ since both are maximal.

Hence $\{p_1, \dots, p_r\}$ is all prime ideals of A.

Since all p_i maximal, they are coprime, so

$$A/\eta \cong \underbrace{A/p_1 \times \dots \times A/p_r}_{\text{product of fields}} \Rightarrow A/\eta \text{ is artinian ring}$$

Since A is noeth, η is finitely gen $\Rightarrow \exists n > 0$ s.t. $\eta^n = 0$

Each η^i/η^{i+1} is a f.g. A/η-module, hence has finite length.

By considering $0 \to \eta^{i+1} \to \eta^i \to \eta^i/\eta^{i+1} \to 0$, we have $\ell(\eta^i) = \ell(\eta^{i+1}) + \ell(\eta^i/\eta^{i+1})$.

$$\Rightarrow \ell(A) = \sum_{i=0}^{n-1} \ell(\eta^i/\eta^{i+1}) < \infty.$$

$\Rightarrow A$ is artinian $\square$

Thm. Every artinian ring A is isomorphic to a direct product of local artinian rings, namely the localizations A_m where m ranges over all maximal ideals. This direct product decomposition is unique up to ordering and isomorphism.

Pf. Let $m_1, \dots m_n$ be the maximal ideals of A. They are all of the prime ideals, and coprime $\Rightarrow$

$$m_1 \cdots m_n = m_1 \cap \dots \cap m_n = \sqrt{0} = \eta.$$

A noeth $\Rightarrow \exists k > 0$ s.t. $\eta^k = 0$.

$$\Rightarrow m_1^k \cap \dots \cap m_n^k = m_1^k \cdots m_n^k = \eta^k = 0.$$

$$\Rightarrow A \cong A/m_1^k \times \dots \times A/m_n^k.$$

Each A/m_i^k is artinian and local.

Localize at m_i. If $j \neq i$, then $(A/m_j^k)_{m_i} = 0$ since
$\exists\, y \in m_j \setminus m_i$ which is both nilpotent & invertible.

Also, the image of m_i in A/m_i^k is its unique maximal ideal,
so $(A/m_i^k)_{m_i} \cong A/m_i^k$.

$\Rightarrow \quad A_{m_i} \cong (A/m_1^k)_{m_i} \times \cdots \times (A/m_n^k)_{m_i} \cong A/m_i^k$.

$\Rightarrow \quad A \cong A_{m_1} \times \cdots \times A_{m_n}$.

Suppose we have $A \cong B_1 \times \cdots \times B_r$, B_i artinian local nonzero rings.

Every prime ideal of $B_1 \times \cdots \times B_r$ is of the form $I_1 \times \cdots \times I_r$
where for some i, $I_i \subset B_i$ is prime & $I_j = B_j$ for $j \neq i$.

Since every prime is maximal, $r = n$.

If we localize at $p_i = I_1 \times \cdots \times I_r$ where $I_i \neq B_i$. Then
$(A_j)_{p_i} = 0$ if $j \neq i$ and $(A_i)_{p_i} \cong A_i$.

$\Rightarrow$ claimed uniqueness. $\square$

Summary: The following are the same:

1. Artinian rings
2. Noetherian rings s.t. every prime ideal is maximal.
3. Rings which are finite length as modules over themselves.
4. Finite direct products of artinian local rings.