Hilbert-Samuel polynomial

A = noeth. local ring, m = maximal ideal
q = m-primary ideal (i.e., $\sqrt{q} = m$) generated by $x_1, \dots, x_s \in A$

Prop. M = finitely gen. A-module, $\mathcal{J}$ stable q-filtration of M. Then:

① M/M_n is finite length module for all $n \geq 0$.
② $\exists$ polynomial $g(x)$ of degree $\leq s$ st. $g(n) = \ell(M/M_n)$ for $n \gg 0$.
Furthermore, $\deg(g(x))$ = order of pole at $t=1$ of $H_{gr_{\mathcal{J}}(M)}(t)$
③ The degree and leading coeff. of $g(x)$ do not depend on choice of stable q-filtration. (only on M and q)

Pf. Set $B := gr_q(A) = \bigoplus_{n \geq 0} q^n/q^{n+1}$

$$N := gr_{\mathcal{J}}(M) = \bigoplus_{n \geq 0} M_n/M_{n+1}.$$

B is noeth. ring, N is finitely gen. B-module.

$\Rightarrow$ $N_n = M_n/M_{n+1}$ is a f.g. $\underset{A/q}{\overset{\shortparallel}{B_0}}$-module for all n.

Since $\sqrt{q} = m$, A/q is artinian $\Rightarrow$ M_n/M_{n+1} is finite length for all n.

We have exact sequences:

$$0 \to M_{n-1}/M_n \to M/M_n \to M/M_{n-1} \to 0$$

$$\Rightarrow \ell(M/M_n) = \sum_{i=0}^{n-1} \ell(M_i/M_{i+1}) < \infty$$

$$\Rightarrow (1-t) \sum_{n \geq 0} \ell(M/M_n) t^n = t \sum_{n \geq 0} \ell(M_n/M_{n+1}) t^n$$

B is generated by degree 1 elements: The images of $x_1, \dots, x_s$ under $\mathfrak{q} \to \mathfrak{q}/\mathfrak{q}^2$.

$\Rightarrow H_N(t) = \sum_{n \geq 0} \ell(M_n/M_{n+1}) t^n$ is a rational function (in t)

w/ denominator $(1-t)^s$

$$\Rightarrow \sum_{n \geq 0} \ell(M/M_n) t^n = \frac{\text{polynomial}(t)}{(1-t)^{s+1}}$$

$\Rightarrow \exists$ polynomial $g(x)$ of degree $\leq s$ s.t. $g(n) = \ell(M/M_n)$ for $n \gg 0$.

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③ Let $M = M_0' \supseteq M_1' \supseteq \cdots$ be another stable $\mathfrak{q}$-filtration

$\Rightarrow \exists$ polynomial $g'(x)$ of degree $\leq s$ s.t. $g'(n) = \ell(M/M_n')$ for $n \gg 0$.

$\exists n_0$ s.t. $M_{n+n_0} \subseteq M_n'$ & $M'_{n+n_0} \subseteq M_n$ for all $n \geq 0$.

$\Rightarrow \ell(M/M_{n+n_0}) \geq \ell(M/M_n')$ for all $n \geq 0$.

& $\ell(M/M'_{n+n_0}) \geq \ell(M/M_n)$

$\Rightarrow$ For $n \gg 0$, $g(n+n_0) \geq g'(n) \geq g(n-n_0)$

Divide by $g(n+n_0)$:

$$1 \geq \frac{g'(n)}{g(n+n_0)} \geq \frac{g(n-n_0)}{g(n+n_0)} \quad \text{for } n \gg 0.$$

As $n \to \infty$ we get $1 \geq \lim_{n \to \infty} \frac{g'(n)}{g(n+n_0)} \geq 1$

$\Rightarrow g'(x)$ and $g(x)$ have same leading coeff. and degree. $\square$

The Hilbert-Samuel polynomial of M (with respect to $\mathfrak{q}$) is denoted $\chi_{\mathfrak{q}}^M(x)$, and is unique polynomial s.t.

$$\chi_{\mathfrak{q}}^{M}(n) = \ell(M/\mathfrak{q}^n M) \quad \text{for } n \gg 0.$$

If $M = A$, write $\chi_{\mathfrak{q}}(x)$ for $\chi_{\mathfrak{q}}^{A}(x)$.

<u>Prop.</u> $\deg \chi_{\mathfrak{q}}(x) = \deg \chi_{\mathfrak{m}}(x)$.

<u>Pf.</u> $\mathfrak{m}$ f.g. & $\sqrt{\mathfrak{q}} = \mathfrak{m} \Rightarrow \exists d$ s.t. $\mathfrak{m}^d \subseteq \mathfrak{q}$

For all $n \geq 0$, $\mathfrak{m}^{dn} \subseteq \mathfrak{q}^n \subseteq \mathfrak{m}^n$

$$\Rightarrow \ell(A/\mathfrak{m}^{dn}) \geq \ell(A/\mathfrak{q}^n) \geq \ell(A/\mathfrak{m}^n)$$

$\Rightarrow$ for $n \gg 0$ $\chi_{\mathfrak{m}}(dn) \geq \chi_{\mathfrak{q}}(n) \geq \chi_{\mathfrak{m}}(n)$

$\Rightarrow \deg \chi_{\mathfrak{m}}(x) = \deg \chi_{\mathfrak{q}}(x)$. $\square$

Summary: (Set $\mathbb{k} = A/\mathfrak{m}$)

$$(1-t)\sum_{n \geq 0} \dim_{\mathbb{k}} (A/\mathfrak{m}^n) t^n = t \sum_{n \geq 0} \dim_{\mathbb{k}} (\mathfrak{m}^n/\mathfrak{m}^{n+1}) t^n$$

$$= t \cdot H_{gr_{\mathfrak{m}}(A)}(t).$$

$\Rightarrow \deg \chi_{\mathfrak{m}}(x) =$ order of pole at $t=1$ of $H_{gr_{\mathfrak{m}}(A)}(t)$

<u>Prop.</u> $x \in A$ nonzerodivisor on M.
Set $M' = M/xM$. <u>Then</u>: $\deg \chi_{\mathfrak{q}}^{M'} \leq \deg \chi_{\mathfrak{q}}^{M} - 1$.

<u>Pf.</u> Set $N = xM$, $N_n = N \cap \mathfrak{q}^n M$

(By Artin-Rees lemma, N_n gives stable $\mathfrak{q}$-filtration on N)

Exact sequence: $0 \to N/N_n \to M/\mathfrak{q}^n M \to M'/\mathfrak{q}^n M' \to 0$

$\Rightarrow$ for $n \gg 0$, $\chi_q^{M'}(n) = \chi_q^{M}(n) - \ell(N/N_n)$

Since x is NZD on M, the map $\begin{array}{l} M \to N \\ m \to xm \end{array}$ is an isomorphism. For $n \gg 0$, $n \mapsto \ell(N/N_n)$ is a polynomial function which has same degree and leading coeff as $\chi_q^{N}(x) = \chi_q^{M}(x)$

$\Rightarrow$ $\deg \chi_q^{M'}(x) \leq \deg \chi_q^{M}(x) - 1$. $\square$