Definition of dimension

A = noeth. local ring, m = maximal ideal, $\mathbb{k} = A/m$

① $\delta(A)$ = least # of generators of q, q is any m-primary ideal
(i.e., $\sqrt{q} = m$)

② $d(A) = \deg \chi_m(x) \leftarrow$ Hilbert–Samuel polynomial
i.e., $\chi_m(n) = \ell(A/m^n)$ for $n \gg 0$

③ $\dim A$ = supremum of the length of any strictly increasing chain of prime ideals in A:

$$\dim A = \sup\{d \mid \exists\ p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_d \subset A\}$$

p_i are primes

<u>Goal</u>: Show $\delta(A) = d(A) = \dim A$.

This is the <u>(Krull) dimension</u> of A

<u>Prop</u>. $\delta(A) \geq d(A)$.

<u>Pf</u>. Let $s = \delta(A)$. $\exists$ m-primary ideal q gen. by s elements.

$\Rightarrow s \geq \deg \chi_q(x) = \deg \chi_m(x)$. $\square$

<u>Prop</u>. $d(A) \geq \dim A$.

<u>Pf</u>. Induction on $d(A)$.

If $d(A) = 0$, then $\chi_m(x)$ is constant

$\Rightarrow \ell(A/m^n)$ is constant for $n \gg 0$.

$\Rightarrow \ell(m^n/m^{n+1}) = 0$ for $n \gg 0$

$\Rightarrow m^n = m^{n+1}$ for $n \gg 0$.

$\Rightarrow m^n = 0$ for $n \gg 0$. (Nakayama's Lemma)

Let p be a prime ideal of A.

Then $m^n = 0 \subseteq p \Rightarrow m \subseteq p. \Rightarrow m = p.$

$\Rightarrow \dim A = 0.$

Now suppose $d(A) > 0$. Pick $p_0 \subsetneq p_1 \subsetneq \cdots \subsetneq p_r$ be a strict chain of prime ideals. We need to show $r \le d(A)$.

We may assume $r > 0$. Let $A' = A/p_0$. (domain)

Let m' be maximal ideal of A' ($=$ image of m)

Let p_i' be image of p_i in A'.

Pick $x \in p_1'$, $x \neq 0$. So x is NZD.

$\Rightarrow d(A'/x) \le d(A') - 1$ (from previous lecture)

Furthermore, for all n, we have surjection

$$A/m^n \twoheadrightarrow A'/(m')^n \Rightarrow \ell(A/m^n) \ge \ell(A'/(m')^n)$$

$$\Rightarrow d(A) \ge d(A').$$

$\Rightarrow d(A'/x) \le d(A) - 1$

By induction, $\dim(A'/x) \le d(A'/x)$.

Claim: for all $i \ge 1$, $p_i'/x \neq p_{i+1}'/x$

(If $p_i'/x = p_{i+1}'/x$, then every element of p_{i+1}' is equal to an element of p_i' plus multiple of x, but $x \in p_i'$, so this contradicts $p_i \neq p_{i+1}$)

$\Rightarrow p_1'/x \subsetneq p_2'/x \subsetneq \cdots \subsetneq p_r'/x$ in A'/x.

$\Rightarrow r - 1 \le \dim(A'/x) \le d(A'/x) \le d(A) - 1$

$\Rightarrow d(A) \ge r.$ $\square$

Note: This implies that $\dim A < \infty$.

Def. Let $p \lhd A$ (A not necessarily local) be a prime. The height of p (height(p)) is maximal length of any strictly increasing chain of prime ideals inside of p.

Note: height(p) = $\dim(A_p)$.

Prop. $\dim A \geq \delta(A)$.

Pf. Let $d = \dim A$. We will construct an ideal $(x_1, \dots, x_d)$ which is m-primary.

We will prove the following statement by induction on i:

(*) If $i \leq d$, $\exists$ sequence $x_1, \dots x_i \in A$ s.t. every prime p that contains $x_1, \dots x_i$ satisfies height(p) $\geq i$.

Base case $i=0$: Vacuous, since height(p) ≥ 0.

Inductive step: Suppose $i > 0$ and $x_1, \dots x_{i-1}$ exist satisfying (*).

A noeth $\Rightarrow$ $(x_1, \dots, x_{i-1})$ has finitely many minimal prime ideals. Let $\{p_1, \dots p_s\}$ be those min. primes of height $= i-1$.

Since $i-1 < d = $ height(m), $m \notin \{p_1, \dots, p_s\}$.

Prime avoidance $\Rightarrow$ $m \not\subseteq p_1 \cup \dots \cup p_s$.

$\Rightarrow \exists x_i \in m$ s.t. $x_i \notin p_1 \cup \dots \cup p_s$.

Claim: $x_1, \dots x_i$ satisfies (*).

Let q be any prime containing $(x_1, \dots x_i)$.

Want to show: $\text{height}(q) \geq i$.

Since $q \supseteq (x_1, \dots x_{i-1})$, it contains some min. prime p of $(x_1, \dots, x_{i-1})$.

Case 1: If $p \in \{p_1, \dots, p_s\}$ then $\text{height}(p) = i-1$

$\Rightarrow x_i \notin p \Rightarrow p \subsetneq q \Rightarrow \text{height}(q) \geq \text{height}(p)+1 = i$ ✓

Case 2: If $p \notin \{p_1, \dots, p_s\} \Rightarrow \text{height}(p) \geq i$

$\Rightarrow \text{height}(q) \geq i$. ✓

Hence $x_1 \dots x_i$ satisfies $(*)$.

Induction done.

So $\exists\ x_1, \dots x_d$ s.t. every prime p containing $x_1 \dots x_d$ has $\text{height}(p) \geq d$.

$\Rightarrow$ m is the only prime containing $x_1, \dots x_d$.

$\sqrt{(x_1, \dots x_d)}$ = intersection of all primes containing $= m$

$\Rightarrow (x_1, \dots x_d)$ is m-primary. $\square$

MAIN THM: $\delta(A) = d(A) = \dim A$.

Def. If $d = \dim A$, and $x_1, \dots x_d$ generate m-primary ideal, then they are called system of parameters (SOP).

Thm (Krull Principal ideal Thm).

Let A = noeth. ring, $x_1, \dots x_r \in A$.

Let p be min. prime of $(x_1, \dots x_r)$, Then $\text{height}(p) \leq r$.

Pf Consider A_p and images of $x_1, \dots, x_r$ under $A \to A_p$.

pA_p is unique maximal ideal of A_p.

$\Rightarrow (x_1, \dots x_r)$ is pA_p-primary ideal.

$\Rightarrow r \geq \dim A_p = \text{height}(p)$. $\square$

Cor. A = noeth. local ring, $x \in m$ NZD.

Then: $\dim(A/x) = \dim A - 1$.

Pf. Let $d = \dim(A/x)$. From before:

$$d(A/x) \leq d(A) - 1.$$

Pick $x_1, \dots x_d \in A$ s.t. images in A/x are SOP.

Let p be any prime containing $(x, x_1, \dots x_d)$.

Then $p/x = m/x \Rightarrow p = m$ b/c p is inverse image of m/x under $A \to A/x$, so $p \supseteq m$.

$\Rightarrow (x, x_1, \dots x_d)$ is m-primary

$\Rightarrow d+1 \geq \dim A \Rightarrow d+1 = \dim A$. $\square$

Let A be any ring. The dimension of A is

$$\dim A = \sup\{d \mid \exists\, p_0 \subsetneq \cdots \subsetneq p_d \subset A\}$$

primes of A.

$\Rightarrow \dim A = \sup\{\dim A_p \mid p \in \operatorname{Spec} A\}$

Note: $\dim A$ need not be finite even if A is noetherian.