Noether normalization

__Lemma.__ $F(x_1,\dots,x_n)$ = homog. poly w/ coefficients in an infinite field $\mathbb{k}$. __Then:__ $\exists\ \lambda_1,\dots\lambda_n$ (all $\neq 0$) $\in \mathbb{k}$ s.t.

$F(\lambda_1,\dots,\lambda_n)\neq 0$.

__Pf.__ Induction on n. If $n=1$, $F(x_1)=x_1^d$, take any nonzero λ_1 ✓.

Now assume $n>1$. Consider F as polynomial in x_n w/ coeff in $\mathbb{k}[x_1,\dots,x_{n-1}]$:

$$F=\sum_{i=0}^{d} F_i(x_1\dots x_{n-1})\,x_n^i$$

Some $F_i\neq 0$, pick one. By induction, $\exists$ nonzero $\lambda_1\dots\lambda_{n-1}$ s.t. $F_i(\lambda_1,\dots,\lambda_{n-1})\neq 0$.

$F(\lambda_1,\dots\lambda_{n-1},x_n)$ is a polynomial in $\mathbb{k}[x_n]$, not zero. Take λ_n to be any nonzero element, not a root of this polynomial. $\square$

__Rmk.__ May assume $\lambda_n=1$: if F has degree d,

$$F(\lambda_1,\dots\lambda_n)=\lambda_n^d\,F\left(\frac{\lambda_1}{\lambda_n},\dots,\frac{\lambda_{n-1}}{\lambda_n},1\right)$$

__Thm__ (Noether normalization). $\mathbb{k}$ = infinite field, A = finitely generated $\mathbb{k}$-algebra. __Then__ $\exists$ $\mathbb{k}$-subalgebra $B\subset A$ s.t. A is integral over B & B is isomorphic to a polynomial ring over $\mathbb{k}$.

Pf. Induction on number n of generators of A.

If $n=0$, $A=\mathbb{k}$, nothing to prove.

Assume $n>0$ and statement holds for algebras gen. by $n-1$ elements.

Let A be generated by $x_1, \dots x_n$.

If $x_1, \dots x_n$ are alg. ind. over $\mathbb{k}$, take $B=A$. ✓

Else, may reorder so that $x_1, \dots, x_i$ are alg. ind. over $\mathbb{k}$, & $x_{i+1}, \dots x_n$ algebraic over $\mathrm{Frac}(\mathbb{k}[x_1, \dots, x_i])$

$\Rightarrow$ x_n is algebraic over $\mathrm{Frac}(\mathbb{k}[x_1, \dots x_i])$

$\Rightarrow$ nonzero polynomial f w/ coeff. in $\mathbb{k}$ s.t. $f(x_1, \dots x_n) = 0$.

Let F = sum of highest degree terms in f.

F is homog (let d = degree)

$\Rightarrow$ $\exists\ \lambda_1, \dots \lambda_{n-1} \in \mathbb{k}$ s.t. $F(\lambda_1, \dots, \lambda_{n-1}, 1) \neq 0$.

Set $x_i' = x_i - \lambda_i x_n$ for $i=1, \dots n-1$,

Set $A' = \mathbb{k}[x_1', \dots, x_{n-1}']$.

<u>Claim</u>: x_n is integral over A'.

$$f(x_1, \dots x_n) = 0 \Rightarrow \underbrace{f(x_1' + \lambda_1 x_n, \dots, x_{n-1}' + \lambda_{n-1} x_n, x_n)}_{} = 0$$

can think of as poly. expression in x_n w/ coeff. in $\mathbb{k}[x_1', \dots x_{n-1}']$

leading term is $F(\lambda_1, \dots, \lambda_{n-1}, 1)\, x_n^d$

Define new variable t, consider monic polynomial.

$$\frac{1}{F(\lambda_1, \dots \lambda_{n-1}, 1)} f(x_1' + \lambda_1 t, \dots x_{n-1}' + \lambda_{n-1} t, t) \in A'[t]$$

x_n is a solution to this polynomial

$\Rightarrow$ x_n is integral over A'.

By induction, $\exists\, B \subset A'$ s.t. A' integral over B
and $B \cong$ poly. ring over $\mathbb{k}$

By transitivity, A is integral over B. $\square$

<u>Rmk</u>. If $x_1, \dots x_n$ generate A as $\mathbb{k}$-algebra proof shows that can always take B to be generated by $\mathbb{k}$-linear combinations of $x_1, \dots x_n$.

In fact, "generic" (or "random") choice of linear combinations suffice.

How many generators does B have?

Answer: $\dim A$