Regular Local Rings

A = noeth. local ring, m = maximal ideal, $\mathbb{k} = A/m$ residue field.

(Zariski) cotangent space of A is

$$m/m^2 \leftarrow \text{vector space over } \mathbb{k}$$

Know: $\dim_{\mathbb{k}}(m/m^2) \geq \dim A$

If $\dim_{\mathbb{k}}(m/m^2) = \dim A$, then A is a regular local ring.

For a general noeth ring A, we call A regular if A_p are regular local rings for all prime ideals p.

Question: If A is regular local ring, is A_p also regular local ring? Yes, see references for proof.

Intuition: Dual space of m/m^2 is "tangent space"

In calculus, given smooth manifold X, point $x \in X$, a tangent vector at x is obtained from smooth curve

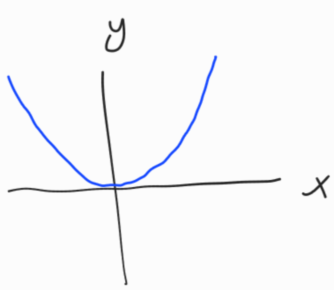
$$\gamma : [0,1] \to X \quad \text{s.t.} \quad \gamma(0) = x$$

Derivative $\gamma'(0)$ is captured by linear part of its Taylor expansion, role played by $(m/m^2)^*$.

Ex. $A = \mathbb{C}[x,y]/(f) \qquad f \neq 0.$

"ring of functions on $\{(\alpha,\beta) \in \mathbb{C}^2 \mid f(\alpha,\beta) = 0\}$"

f is nonzero divisor $\Rightarrow \dim A = 1$



(1) $f = x^2 - y$ "parabola"

$m = (x,y)$

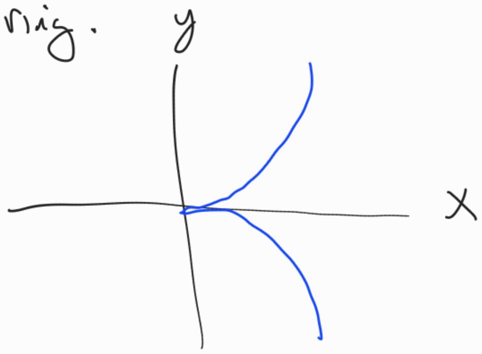
$m^2 = (x^2, xy, y^2) \supseteq (y)$ since $x^2 = y$ in A_m

m/m^2 is spanned by $\{x\}$

Know: $\dim_{\mathbb{C}} m/m^2 \geq 1$, so actually equal to 1.

$\Rightarrow A_m$ is regular local ring.

② $f = x^3 - y^2$ "cusp"



$m = (x, y)$

$m^2 = (x^2, xy, y^2)$

Claim: $\{x, y\}$ is linearly independent in m/m^2.

$\Rightarrow \dim_{\mathbb{C}} m/m^2 = 2 \Rightarrow A_m$ is not regular

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Prop. A = d-dim noeth local ring, m = maximal ideal, $\mathbb{K} = A/m$.

The following are equivalent:

① A is regular local ring, i.e., $\dim_{\mathbb{K}}(m/m^2) = d$

② m can be generated by d elements

③ $\exists$ isomorphism of graded rings $gr_m(A) \cong \mathbb{K}[x_1, \ldots x_d]$ $(\deg x_i = 1)$

Pf. ① $\Rightarrow$ ② If $a_1, \ldots a_d \in m$ go to basis in m/m^2

then $a_1, \ldots a_d$ generate m by Nakayama's lemma.

② $\Rightarrow$ ③ Pick generators $a_1, \ldots a_d \in m$. Define

$\alpha : \mathbb{K}[x_1, \ldots x_d] \to gr_m(A)$ $\quad \bar{a}_i$ = image of a_i in m/m^2

$\alpha(f(x_1, \ldots x_d)) = f(\bar{a}_1, \ldots \bar{a}_d)$

α is surjective since $a_1, \ldots a_d$ generate m

Suppose $f \in \ker \alpha$. May assume f is homogeneous of degree s.

$f \in \ker \alpha \Rightarrow f(a_1, \ldots a_d) \in m^{s+1}$

$\Rightarrow$ $f \in \mathbb{k}[x_1, \dots, x_d]$ is a zerodivisor $\Rightarrow f = 0$.

$\Rightarrow$ α is isomorphism.

③ $\Rightarrow$ ① $m/m^2 \cong \mathbb{k}$-span of $x_1, \dots x_d$, which has dim d. □

Cor. Every regular local ring is an integral domain.

Pf. Let A be regular local w/ max'l ideal m.

Pick $x, y \in A$ nonzero. Note: $\bigcap_n m^n = 0$ by Nakayama

$[\, m \bigcap_n m^n = \bigcap_n m^n \,]$

So $\exists\, r, s$ s.t. $x \in m^r \setminus m^{r+1}$

$y \in m^s \setminus m^{s+1}$

image of x in $gr_m(A)_r$ is nonzero

y in $gr_m(A)_s$ is nonzero

$\exists$ isomorphism $gr_m(A) \cong \mathbb{k}[x_1, \dots x_d]$ ← domain

image of $xy \in gr_m(A)_{r+s}$ is nonzero.

$\Rightarrow xy \neq 0$. □

Rmk. Every regular local ring is a UFD.

By previous results, an artinian local ring is dim 0

so A is regular $\Leftrightarrow$ $m/m^2 = 0 \Leftrightarrow m = m^2$

By Nakayama $\Leftrightarrow m = 0$

$\Leftrightarrow$ being a field.

Hence regular ring of dim 0 is same thing as finite direct product of fields