Krull topology

Γ = abelian group, $\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \dots$ subgroups.

Krull topology on Γ is coarsest topology s.t. cosets $g + \Gamma_i$ are open for all $g \in \Gamma$ and all i.

Note: every coset is closed (complement is union of cosets, hence open)

Prop. Every open set is a union of cosets.

Pf. We need to show: intersection of two unions of cosets is again union of cosets. Since $(\bigcup_i S_i) \cap (\bigcup_j T_j) = \bigcup_{i,j} S_i \cap T_j$, suffices to consider intersection of two cosets. Pick $g + \Gamma_i$, $h + \Gamma_j$ ($i \geq j$ WLOG)

Claim: Have $(g+\Gamma_i) \cap (h + \Gamma_j) = \begin{cases} \emptyset \\ g + \Gamma_i & \text{if nonempty} \end{cases}$

Suppose nonempty. Pick $x \in (g+\Gamma_i) \cap (h+\Gamma_j)$. So $\exists g' \in \Gamma_i$, $\exists h' \in \Gamma_j$ s.t.

$x = g + g' = h + h'$. Since $\Gamma_i \subseteq \Gamma_j$, $g' - h' \in \Gamma_j$

$\Rightarrow g - h \in \Gamma_j$. (Since $g - h = h' - g'$)

$\Rightarrow g + \Gamma_i \subseteq g + \Gamma_j = h + \Gamma_j$

$\Rightarrow (g+\Gamma_i) \cap (h + \Gamma_j) = g + \Gamma_i$. $\square$

A Cauchy sequence is a sequence $g_1, g_2, \dots \in \Gamma$ s.t. for all open neighborhoods U of 0, $\exists n$ s.t. if $i, j > n$ then $g_i - g_j \in U$ (i.e., $g_i - g_j \in U$ if $i, j \gg 0$)

Given Cauchy sequences (g_i), (g_i'), define $(g_i) \sim (g_i')$ if for all open neighborhood U of 0, have $g_i - g_i' \in U$ for $i \gg 0$.

[Note: can replace "all open neighborhoods" w/ "all subgroups Γ_i"]

Prop. The set of Cauchy sequences is a group under componentwise addition. The set of Cauchy sequences equivalent to 0 sequence is subgroup.

Pf. Let $(g_i), (h_i)$ be Cauchy sequences. Pick subgroup Γ_k. Then for $i, j \gg 0$, $g_i - g_j \in \Gamma_k$ & $h_i - h_j \in \Gamma_k$.

$$\Rightarrow (g_i + h_i) - (g_j + h_j) \in \Gamma_k$$

$\Rightarrow$ $(g_i + h_i)_i$ is a Cauchy sequence

Similarly, can show $(-g_i)$ is a Cauchy sequence.

Second statement proven similarly, (or use next result) $\square$

Let $\mathcal{C}(\Gamma)$ = group of Cauchy sequences (depends on filtration of subgroups)

$\mathcal{C}_0(\Gamma)$ = Subgroup of Cauchy sequences $\sim (0)$.

Define homomorphism $\Phi : \mathcal{C}(\Gamma) \to \hat{\Gamma}$:

Pick Cauchy sequence $(g_i) \in \mathcal{C}(\Gamma)$. For each Γ_k, we have $g_i - g_j \in \Gamma_k$ for $i, j \gg 0$, i.e., $g_i + \Gamma_k = g_j + \Gamma_k$ for $i, j \gg 0$.

let c_k denote this coset.

Define $\boxed{\Phi((g_i)) = (c_k)}$

Well-defined: c_k & c_{k+1} can be represented by same element g_i (take $i \gg 0$) so c_{k+1} becomes c_k modulo Γ_k.

Prop. Φ is surjective, $\ker \Phi = \mathcal{C}_0(\Gamma)$.

Hence: $\mathcal{C}(\Gamma) / \mathcal{C}_0(\Gamma) \cong \hat{\Gamma}$.

Pf. If $\Phi((g_i)) = 0$, then cosets c_k are trivial for all k.

$\Rightarrow \forall k$ we have $g_i \in \Gamma_k$ for $i \gg 0$. $\Rightarrow (g_i) \sim (0) \Rightarrow (g_i) \in \mathcal{C}_0(\Gamma)$.

Using reverse implications, see that $\mathcal{C}_0(\Gamma) \subseteq \ker \Phi$.

__Surjectivity:__ Pick sequence $(g_i + \Gamma_i) \in \hat{\Gamma}$.

Pick subgroup Γ_k. If $i, j \geq k$, then

$g_i \equiv g_k \pmod{\Gamma_k}$

$g_j \equiv g_k \pmod{\Gamma_k}$

$\Rightarrow g_i - g_j \in \Gamma_k$. $\Rightarrow (g_i)$ is a Cauchy sequence.

By construction, $\Phi((g_i)) = (g_i + \Gamma_i)$. $\square$

__Note:__ $\hat{\Gamma}$ depends only on Krull topology defined by decreasing filtration of subgroups.

$\Rightarrow$ If $\Gamma_1 \supseteq \Gamma_2 \supseteq \dots$ & $\Gamma_1' \supseteq \Gamma_2' \supseteq \dots$ give same topology, then completions are same.

Given homomorphism $f: \Gamma \to \Gamma'$. (suppose both have decreasing filtrations)

__Note:__ If $f(\Gamma_i) \subseteq \Gamma_i'$, then

get $\hat{f}: \hat{\Gamma} \to \hat{\Gamma}'$.

More generally, if f is continuous wrt Krull topology, then applying f pointwise to Cauchy sequence in Γ gives another Cauchy sequence $\Rightarrow$ get homomorphism

$$\hat{f}: \underset{\hat{\Gamma}}{\underset{\cong}{\mathcal{C}(\Gamma)/\mathcal{C}_0(\Gamma)}} \to \underset{\hat{\Gamma}'}{\underset{\cong}{\mathcal{C}(\Gamma')/\mathcal{C}_0(\Gamma')}}.$$

Functorial: given $g: \Gamma' \to \Gamma''$, continuous

then $\hat{g}\hat{f} = \widehat{gf}$