Discrete valuation rings

K = field, a discrete valuation is function $v: K \setminus 0 \to \mathbb{Z}$ st. for all $x, y \in K \setminus 0$, we have

① $v(xy) = v(x) + v(y)$, and

② $v(x+y) \geq \min(v(x), v(y))$

Convention: $v(0) = \infty$

Note: ① $\Rightarrow$ $v(1) = 0$ & $v(1/x) = -v(x)$ if $x \neq 0$.

Define $A = \{0\} \cup \{x \in K \mid v(x) \geq 0\}$ valuation ring

$m = \{0\} \cup \{x \in K \mid v(x) > 0\}$

Note: A is local w/ maximal ideal m:

if $x \in A \setminus m$, then $v(x) = 0$, so $v(1/x) = 0$ and $1/x \in A$

Examples ① $K = \mathbb{Q}$, p prime number

Every rational number is of the form $p^a x$

where numerator/denominator of x prime to p, $a \in \mathbb{Z}$.

Define v_p by $v_p(p^a x) = a$.

Valuation ring is $\mathbb{Z}_{(p)}$ w/ maximal ideal (p)

② $K = \mathbb{Q}_p$ = p-adic numbers.

Every p-adic number is of the form $\sum_{n \geq N} a_n p^n$ where $0 \leq a_i < p$

If $a_N \neq 0$, define valuation to be N.

valuation ring is p-adic integers $\mathbb{Z}_p$.

(3) $\Bbbk$ = any field, $K = \Bbbk(x)$ = rational functions in x.

Let $f \in \Bbbk[x]$ irreducible polynomial

Every rational function is of form $f^a \dfrac{g_1}{g_2}$ polynomials prime to f

$a \in \mathbb{Z}$

Define v_f by $v_f(f^a \frac{g_1}{g_2}) = a$

valuation ring is $\Bbbk[x]_{(f)}$

(4) $K = \Bbbk((x))$ Laurent series in x

For Laurent series $\sum_{n \geq N} a_n x^n$ w/ $a_N \neq 0$, define its valuation to be N. The valuation ring is $\Bbbk[[x]]$.

A local domain A is a <u>discrete valuation ring (DVR)</u> if $\exists$ discrete valuation v on $\mathrm{Frac}(A)$ so that A is its valuation ring.

<u>Prop</u>. Every DVR is noeth. & has dimension 1.

<u>Pf</u>. Let A be a DVR, let v be discrete valuation on $K = \mathrm{Frac}(A)$. If $v(x) = v(y)$, then $v(xy^{-1}) = 0$, (and $x, y \neq 0$)

so $x/y \in A$. Hence if $v(x) = v(y)$ then $(x) = (y)$

Let I be nonzero ideal of A, let $k = \min\{v(x) \mid x \in I \setminus 0\}$

Then I contains every element of valuation k.

$\implies I = m^k$.

Hence A satisfies ACC on ideals since they're all powers of m.

$\Rightarrow$ every prime ideal is either 0 or m.

Since v is surjective onto $\mathbb{Z}$, so $m \neq 0$

$\Rightarrow \dim A = 1$. $\square$

Prop. Let A be noeth. local domain of dim 1.

Let m = maximal ideal, $\mathbb{k} = A/m$. $K = \mathrm{Frac}(A)$.

TFAE:

1. A is a DVR.
2. A is normal.
3. m is principal ideal
4. A is regular local ring, i.e., $\dim_{\mathbb{k}}(m/m^2) = 1$
5. Every nonzero ideal is a power of m.
6. $\exists\, x \in A$ s.t. every nonzero ideal is of the form (x^k) for some $k \geq 0$.

pf. Rmk: Since $\dim A = 1$, only prime ideals are $0, m$.

So for any nonzero ideal I, have $\sqrt{I} = m$.

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① $\Rightarrow$ ② Let $v: K \setminus 0 \to \mathbb{Z}$ be discrete valuation s.t.

A = valuation ring of v.

Pick $x \in K$ integrally closed over A. Have equation

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \quad \text{for } a_1, \ldots, a_n \in A$$

If $v(x) \geq 0$, then $x \in A$, done. Else, $v(x) < 0$, so $v(1/x) > 0$, and $1/x \in A$. Multiply equation by $\frac{1}{x^{n-1}}$:

$$x + \underbrace{(a_1 + \cdots + a_n 1/x^{n-1})}_{\in A} = 0 \implies x \in A \to\leftarrow$$

$\implies x \in A$ so A is normal.

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② $\implies$ ③ Pick nonzero $a \in m$. Then $\sqrt{(a)} = m$, so

$\exists$ minimal positive n s.t. $m^n \subseteq (a)$. Pick $b \in m^{n-1} \setminus (a)$

set $x = \frac{a}{b} \in K$. For any $y \in m$, we $yb \in m^n \subseteq (a)$

and hence $\frac{y}{x} = \frac{by}{a} \in A$

$\implies x^{-1} m \subseteq A$, and $x^{-1} m$ is an ideal.

Claim: $x^{-1} m = A$.

If not, then $x^{-1} m \subseteq m$. $\implies x^{-1}$ is integral over A.

$\implies x^{-1} \in A$. $\implies \frac{b}{a} \in A \implies b \in (a) \to\leftarrow$

$\implies (x) \subseteq m$, but also $m \subseteq (x)$, so $m = (x)$.

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③ $\iff$ ④ Nakayama's Lemma.

③ $\implies$ ⑤ Let I = nonzero proper ideal of A.

$\bigcap_n m^n = 0 \implies \exists$ largest n s.t. $I \subseteq m^n$

$\implies I \nsubseteq m^{n+1}$, can pick $y \in I \setminus m^{n+1}$.

Write $m = (x)$ (by (3)) $\implies \exists a \in A$ s.t. $y = ax^n$

Since $y \notin m^{n+1}$, a is a unit

$\Rightarrow x^n \in I \Rightarrow m^n \subseteq I \Rightarrow I \supseteq m^n$.

(5) $\Rightarrow$ (6) Since $m \neq 0$, have $m \neq m^2$ by Nakayama's Lemma.

Pick $x \in m \setminus m^2$. By (5), $\exists k$ s.t. $(x) = m^k$

By choice of x, $k=1$, so $m = (x)$.

$\Rightarrow$ Every nonzero ideal is of the form (x^k).

(6) $\Rightarrow$ (1) We have $(x) = m$, so $(x^k) \neq (x^{k+1})$ by Nakayama.

Given nonzero $a \in A$, $\exists$ unique k s.t. $(a) = (x^k)$.

Define $v(a) = k$.

Pick $a, a' \in A$ nonzero. We have

$v(aa') = v(a) + v(a')$ since $(aa') = (a)(a') = (x^{v(a)})(x^{v(a')})$

$= (x^{v(a)+v(a')})$

If $a + a' \neq 0$, then

$(a+a') \subseteq (a, a') = (x^{v(a)}, x^{v(a')}) = (x^{\min(v(a), v(a'))})$

$\Rightarrow v(a+a') \geq \min(v(a), v(a'))$,

Given nonzero $a, b \in A$, define $v\left(\frac{a}{b}\right) = v(a) - v(b)$.

Given nonzero $a', b' \in A$, have

$$v\left(\frac{a}{b} \cdot \frac{a'}{b'}\right) = v(aa') - v(bb')$$
$$= v(a) + v(a') - v(b) - v(b')$$
$$= v\left(\frac{a}{b}\right) + v\left(\frac{a'}{b'}\right) \quad \checkmark$$

Suppose $\frac{a}{b} + \frac{a'}{b'} \neq 0$. Then

$$v\left(\frac{a}{b}+\frac{a'}{b'}\right) = v\left(\frac{ab'+a'b}{bb'}\right)$$

$$= v(ab'+a'b) - v(b) - v(b')$$

$$\geq \min\big(v(a)+v(b'),\ v(a')+v(b)\big) - v(b) - v(b')$$

$$= \min\big(v(a)-v(b),\ v(a')-v(b')\big)$$

$$= \min\left(v\left(\frac{a}{b}\right),\ v\left(\frac{a'}{b'}\right)\right) \quad \checkmark$$

Note: $v(x)=1$, so v is surjective.

$\Rightarrow$ v is a discrete valuation

Finally, pick nonzero $a,b \in A$, suppose $v\left(\frac{a}{b}\right) \geq 0$

i.e, $v(a) \geq v(b)$. $\Rightarrow$ $(a) = (x^{v(a)}) \subseteq (x^{v(b)}) = (b)$

Hence $\exists c \in A$ s.t. $a = bc$

but then $\frac{a}{b} = c \in A$

$\Rightarrow$ A is valuation ring of v. $\square$

Cor. If A is DVR w/ maximal ideal m, then m-adic completion $\hat{A}$ is also DVR.