Integral dependence

$A \subset B$ subring

Def. $x \in B$ is <u>integral over A</u> if $\exists\, n > 0$ and $a_1, \dots, a_n \in A$

s.t.

$$x^n + a_1 x^{n-1} + \dots + a_0 = 0$$

Let $A[x]$ = subring of B generated by A, x.

Prop. TFAE:

1. x is integral over A.
2. $A[x]$ is a finitely generated A-module
3. $\exists$ subring $C \subset B$ that contains $A[x]$ s.t. C is a finitely gen. A-module
4. $\exists$ faithful $A[x]$-module M which is finitely gen. as A-module.

faithful $\to$ $\operatorname{ann}_{A[x]} M = 0$

Pf. (1) $\Rightarrow$ (2) For any $N \geq n$, we can write

$$x^{N-n}(x^n + a_1 x^{n-1} + \dots + a_0) = 0$$

so x^N can be written as A-linear comb. of smaller powers of x.
As A-module, $A[x]$ is gen. by $\{1, x, x^2, \dots\}$, so $\{1, x, \dots, x^{n-1}\}$ suffices.

(2) $\Rightarrow$ (3) Take $C = A[x]$

(3) $\Rightarrow$ (4) Take $M = C$. (note $1 \in C$ not annihilated by anything)

(4) $\Rightarrow$ (1) Multiplication by x gives A-linear map $\varphi : M \to M$.
Cayley-Hamilton $\Rightarrow$ $\exists\, a_1, \dots, a_n$ s.t.

$$(\varphi^n + a_1 \varphi^{n-1} + \dots + a_n)(m) = 0 \quad \forall m \in M$$

$\Rightarrow x^n + a_1 x^{n-1} + \cdots + a_n \in \mathrm{ann}_{A[x]} M$

M faithful $\Rightarrow x^n + a_1 x^{n-1} + \cdots + a_n = 0 \Rightarrow x$ is integral over A. $\square$

<u>Cor.</u> If x, y integral over A, then so is $x \pm y$, and xy.

So, set of integral elements over A is a subring of B.

<u>Pf.</u> Pick $x, y \in B$ integral over A.

$\Rightarrow y$ integral over $A[x]$

$\Rightarrow A[x,y]$ is f.g. $A[x]$-module.

$\Rightarrow A[x,y]$ is f.g. A-module (since $A[x]$ is f.g. A-module)

Since $A[x+y] \subset A[x,y]$, $x+y$ integral over A.

$A[x-y] \subset A[x,y]$, $x-y$ ———

$A[xy] \subset A[x,y]$, xy ——— $\square$

<u>Integral extensions</u> Given $A \subset B$, let $\bar{A}$ = set of integral elements over A. (<u>integral closure</u> of A in B)

If $\bar{A} = A$, then A is <u>integrally closed</u> in B.

If $\bar{A} = B$, then $A \subset B$ is an <u>integral extension</u>.

Given $f: A \to B$ any ring map, extend definition by considering subring $f(A) \subset B$

<u>Prop.</u> Suppose $A \subseteq B \subseteq C$ s.t. B is integral over A & C is integral over B. Then C is integral over A.

In particular, for $A \subset B$, $\bar{\bar{A}} = \bar{A}$.

Pf. Pick $x \in C$. $\exists\, b_1, \dots b_n$ s.t.

$$x^n + b_1 x^{n-1} + \dots + b_n = 0.$$

Let $B' = A[b_1, \dots, b_n] \subset B$. x is integral over B'.

$\Rightarrow$ $B'[x]$ is f.g. B'-module.

Each of $b_1, \dots b_n$ is integral over A. $\Rightarrow$ B' is f.g. A-module.

$\Rightarrow$ $B'[x]$ is f.g. A-module

$A[x] \subset B'[x]$ $\Rightarrow$ x integral over A.

For last statement, consider $A \subset \bar{A} \subset \bar{\bar{A}}$ $\square$

Prop. Let $A \subseteq B$ be integral. If $I \subseteq B$ ideal, then

$A/(A \cap I) \subseteq B/I$ is integral.

Pf. Given $x \in B/I$, lift it to $y \in B$.

Take integrality equation for y, reduce it modulo I. $\square$

Prop. Let $S \subseteq A$ be multiplicative subset. If $A \subseteq B$ integral,

so is $S^{-1}A \subseteq S^{-1}B$.

Pf. Pick $\frac{x}{s} \in S^{-1}B$. x is integral over A $\Rightarrow$

$\exists\, a_1, \dots, a_n \in A$ s.t. $x^n + a_1 x^{n-1} + \dots + a_n = 0$.

Divide by s^n:

$$\left(\frac{x}{s}\right)^n + \frac{a_1}{s} \cdot \left(\frac{x}{s}\right)^{n-1} + \dots + \frac{a_n}{s^n} = 0$$

$\Rightarrow$ x/s is integral over $S^{-1}A$. $\square$

Special case: A = domain, B = field of fractions.

Then $\bar{A}$ = normalization of A, denote by $\tilde{A}$.

Def. A domain is normal if it is equal to its normalization.

Prop. A UFD is normal.

Pf. Let A be a UFD, $B = \mathrm{Frac}(A)$

Pick $\frac{x}{y} \in B$, $x, y \in A$. May assume x, y have no common factors in their prime factorization.

Suppose $\frac{x}{y}$ integral over A so $\exists\, a_1, \dots a_n \in A$ s.t.

$$\left(\frac{x}{y}\right)^n + a_1 \left(\frac{x}{y}\right)^{n-1} + \dots + a_n = 0 .$$ Multiply by y^n:

$$x^n + a_1 y x^{n-1} + \dots + a_n y^n = 0$$

$$\Rightarrow \quad x^n = -y(a_1 x^{n-1} + \dots + a_n y^{n-1})$$

$\Rightarrow$ y divides x^n $\Rightarrow$ y is a unit, so $\frac{x}{y} \in A$. $\square$

Normality is a local property:

Prop. Let A be a domain. TFAE:

1. A is normal.
2. A_p is normal for all $p \in \mathrm{Spec}\, A$.
3. A_m is normal for all maximal ideals $m \subset A$.

Pf. Let $B = \mathrm{Frac}\, A$. Note: $B = \mathrm{Frac}(A_p)$ for any $p \in \mathrm{Spec}\, A$.
i.e., $B_p = B$.

Consider inclusion map $f : A \to \tilde{A}$.

A normal $\Leftrightarrow$ f is surjective.
From before, this is a local property. $\square$