Noetherian modules

A = ring, M = A-module.

M is <u>noetherian</u> if every submodule of M is finitely generated.

A is <u>noetherian</u> if noeth. as module over itself, i.e., every ideal is finitely generated.

M satisfies <u>ascending chain condition</u> (ACC) if, given any chain of submodules $M_1 \subseteq M_2 \subseteq \cdots$ we have $M_n = M_{n+1}$ for $n \gg 0$. i.e., the chain <u>stabilizes</u>.

<u>Prop.</u> TFAE: (1) M is noetherian

(2) M satisfies ACC

(3) Every nonempty set of submodules of M has a maximal element.

<u>Pf.</u> (1) $\Rightarrow$ (2): Let $M_1 \subseteq M_2 \subseteq \cdots$ be given. Let $M' = \bigcup_n M_n$ submodule of M, hence f.g. Let $m_1, \ldots m_n$ be generators. They must belong to some M_n, but then $M' = M_n = M_{n+1} = \cdots$

(2) $\Rightarrow$ (3): Let S be nonempty set of submodules of M. Suppose it has no maximal element. We can create infinite increasing chain of submodules: Pick any $M_1 \in S$. Next, assuming $M_1 \subsetneq \cdots \subsetneq M_n$ given, M_n is not maximal, so $\exists M_{n+1} \supsetneq M_n$ also in S. Continue to get chain.

(3) $\Rightarrow$ (1) Suppose M has non f.g. submodule. i.e., $\exists$ sequence $x_1, x_2, \ldots \in M$ s.t. x_i is not in submodule generated by $x_1, \ldots, x_{i-1}$. Let M_i = submodule gen. by $x_1, \ldots x_i$. But then $S = \{M_1, M_2, \ldots\}$ has no maximal element. $\square$

Prop. Let $0 \to M_1 \to M_2 \xrightarrow{f} M_3 \to 0$ be short exact sequence of A-modules. Then M_2 noeth $\iff$ M_1 & M_3 are noeth.

pf. Suppose M_2 is noeth.

Every chain of submodules of M_1 is also a chain of submodules of M_2 hence stabilizes $\Rightarrow$ M_1 noeth.

Given chain of submodules in M_3, its inverse image in M_2 is a chain and stabilizes $\Rightarrow$ original chain stabilizes $\Rightarrow$ M_3 noeth.

Now suppose M_1 & M_3 are noeth,

Note: Given $N \subseteq N'$ submodules of M_2, we have $N = N' \iff$ $N \cap M_1 = N' \cap M_1$ & $f(N) = f(N')$.

(For any $x \in N' \setminus N$, either $x \in M_1$ or x represents nontrivial coset of M_1.

$\hookrightarrow$ violates $N \cap M_1 \overset{\shortparallel}{} N' \cap M_1$ $\quad$ $\hookrightarrow$ violates $f(N) = f(N')$)

Given $N_1 \subseteq N_2 \subseteq \cdots$ in M_2, both

$N_1 \cap M_1 \subseteq N_2 \cap M_2 \subseteq \cdots$ & $f(N_1) \subseteq f(N_2) \subseteq \cdots$ stabilize

$\Rightarrow$ $N_1 \subseteq N_2 \subseteq \cdots$ stabilizes. $\square$

Cor. Finite direct sum of noeth. modules is noeth.

pf. Consider $0 \to M \to M \oplus N \to N \to 0$. $\square$

Cor. If A is noeth ring, then every f.g. A-module is noeth.

pf. M is f.g. $\iff$ $\exists$ surjection $A^{\oplus n} \twoheadrightarrow M$ for some n.

By previous cor, $A^{\oplus n}$ is noeth, hence so is M. $\square$

Noetherian rings

Prop. If A is noeth, so is A/I for any ideal I.

Prop. If A is noeth., $S \subseteq A$ mult. subset, Then $S^{-1}A$ is noeth.

pf. Every ideal of $S^{-1}A$ is of the form $S^{-1}I$ for some ideal $I \subseteq A$. I f.g $\Rightarrow$ $S^{-1}I$ f.g. $\square$

$I \subseteq A$ ideal, a <u>minimal prime</u> p of I is a prime that contains I & is minimal (via inclusion) w.r.t. this property

Prop. If A is noeth, then set of minimal primes of I is finite.

pf. Suppose not. Then the set of ideals w/ infinitely many minimal primes is nonempty; hence has a maximal element, call it J. J not prime, so $\exists x,y \in A$ s.t. $xy \in J$, but $x \notin J$, $y \notin J$. Let p be min. prime of J. Then $xy \in p$ $\Rightarrow$ $x \in p$ or $y \in p$. $\Rightarrow$ $p \supseteq (J,x)$ or $p \supseteq (J,y)$

$\Rightarrow$ p min prime of (J,x) or min prime of (J,y).

By definition, (J,x) & (J,y) have finitely many primes. $\rightarrow\leftarrow$ $\square$

An ideal I is <u>irreducible</u> if: $I = J_1 \cap J_2 \Rightarrow I = J_1$ or $I = J_2$ for any ideals J_1, J_2.

Prop. Every ideal in noeth. ring can be written as finite intersection of irreducible ideals.

Pf. Suppose false, so set of ideals failing this property is nonempty hence has a maximal element I.

I not irreducible $\Rightarrow \exists J_1, J_2 \neq I$ s.t. $I = J_1 \cap J_2$

$\Rightarrow J_1, J_2$ are finite intersections of irreducible ideals.

$\rightarrow\leftarrow$ since we can substitute into $I = J_1 \cap J_2$. $\square$

Rmk. If I irreducible, then $V(I) \subset \operatorname{Spec} A$ is an irreducible space. Hence every closed subset of $\operatorname{Spec} A$ can be written as finite union of irreducible closed subsets if A is noetherian.

Def. Let $q \subset A$ be proper ideal. Then q is primary if, for all $x, y \in A$, $xy \in q \Rightarrow x \in q$ or $y^n \in q$ for some $n > 0$.

i.e., Every zero divisor of A/q is nilpotent.

Prop. If q primary, then $\sqrt{q}$ is prime.

Pf. Suppose $xy \in \sqrt{q}$. Then $\exists\, n > 0$ s.t. $(xy)^n \in q$.

$\Rightarrow x^n \in q$ ($\Rightarrow x \in \sqrt{q}$) or $\exists\, m > 0$ s.t. $(y^n)^m \in q$

$\Rightarrow$ ($y \in \sqrt{q}$). $\square$

Prop. If A is noeth, every irreducible ideal I is primary, in particular, $\sqrt{I}$ is prime.

Pf. Pick $x, y \in A/I$ s.t. $xy = 0$. Consider chain

$$\operatorname{Ann}(y) \subseteq \operatorname{Ann}(y^2) \subseteq \operatorname{Ann}(y^3) \subseteq \cdots \text{ in } A/I.$$

A/I noeth $\Rightarrow$ chain stabilizes, so $\exists n$ s.t. $\mathrm{Ann}(y^n) = \mathrm{Ann}(y^{n+1})$.

Claim. $(y^n) \cap (x) = 0$.

Pick $z \in (y^n) \cap (x)$, so $\exists a, b \in A/I$ s.t.

$$z = ay^n = bx.$$

$\Rightarrow \quad yz = bxy = 0 \Rightarrow \quad ay^{n+1} = 0 \Rightarrow a \in \mathrm{Ann}(y^{n+1}) = \mathrm{Ann}(y^n)$

and so $z = ay^n = 0$.

Let $\tilde{x}, \tilde{y} \in A$ be representatives for $x, y \in A/I$.

$\Rightarrow \left((\tilde{y}^n) + I \right) \cap \left((x) + I \right) = I$.

I irreducible so either $\tilde{y}^n \in I \Rightarrow y$ nilpotent in A/I

or $\tilde{x} \in I \Rightarrow x = 0$

$\Rightarrow$ I primary. $\square$

Ex. Let $A = \mathbb{Q}[x, y]$. $m = (x, y)$.

Then $m^2 = (x^2, xy, y^2)$ is primary, but

$(x^2, xy, y^2) = (x, y^2) \cap (x^2, y)$ so is reducible.