Math 184, Winter 2022
Homework 1
Due: Friday, Jan. 14 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences.

(1) Prove that every polynomial in x can be written as a linear combination of the polynomials
$$1, 2x-1, (2x-1)^2, (2x-1)^3, (2x-1)^4, \dots.$$

(2) How many ways are there to list the letters of the word MATHEMATICIAN?

(3) Let $n \geq 2$ be an integer. Define the following sets:
$$A = \{S \subseteq [n] \mid 1 \in S \text{ and } 2 \in S\},$$
$$B = \{S \subseteq [n] \mid |\{1,2\} \cap S| = 1\}.$$
Find formulas for the size of each set.

(4) (a) How many triples (A, B, C) of subsets of $[n]$ satisfy $A \subseteq C$ and $B \subseteq C$?
(b) How many triples (A, B, C) of subsets of $[n]$ satisfy $A \cap B = \emptyset$ and $B \cap C \neq \emptyset$?

(5) Let n be a positive integer.

Define A_n to be the set of sequences (of any length) whose entries are either 1 or 2 and such that the sum of the entries is $n+1$. For example, $|A_3| = 5$ and consists of the following sequences:
$$(1,1,1,1), (2,2), (2,1,1), (1,2,1), (1,1,2).$$

Let B_n be the set of subsets $S \subseteq [n]$ with no consecutive elements, i.e., if $i \in S$, then $i+1 \notin S$. For example, $|B_3| = 5$ and consists of the following subsets:
$$\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}.$$

(a) Find a bijection $f \colon A_n \to B_n$ (along with an inverse $g \colon B_n \to A_n$). You don't need to explain why they are inverses, but your description of f and g should be clear and detailed enough so that it is obvious.
(b) Explain how the elements of A_3 and B_3 are matched up by the bijection you gave in (a).
(c) What does your bijection do to the sequence $(1,2,1,2,2,1,1,2,1)$?

1. Extra practice problems (don't turn in)

(6) How many integers are there between 10000 and 99999 in which all digits are different?

(7) How many pairs (A, B) of subsets of $[n]$ satisfy $A \cup B = [n]$?

(8) Let n and k be positive integers. Show that the number of k-tuples $(X_1, \ldots, X_k)$, where each X_i is a subset of $[n]$, and $X_1 \cap X_2 \cap \cdots \cap X_k = \emptyset$ (i.e., there is no element which is in all of the X_i) is $(2^k - 1)^n$.

For example, when $k = 2$ and $n = 2$, here are the 9 2-tuples:

$$\begin{array}{rrr} (\emptyset, \emptyset) & (\emptyset, \{1\}) & (\emptyset, \{2\}) \\ (\emptyset, \{1,2\}) & (\{1\}, \emptyset) & (\{2\}, \emptyset) \\ (\{1,2\}, \emptyset) & (\{1\}, \{2\}) & (\{2\}, \{1\}). \end{array}$$

(9) Let n be a positive integer.

Define A_n to be the set of (finite) sequences whose entries are either 1 or 2 and such that the sum of the entries is $n - 1$ (the length of the sequence is not predetermined)

Define B_n to be the set of sequences whose entries are odd positive integers and such that the sum of the entries is n.

Describe a bijection between A_n and B_n.