Math 184, Winter 2022
Homework 2
Due: Friday, Jan. 21 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences.

(1) Consider the equation

$$x_1 + x_2 + \cdots + x_8 = 44.$$

For each of the following conditions, how many solutions are there? (Each part is an independent problem, don't combine the conditions.)

(a) The x_i are positive even integers.
(b) The x_i are positive odd integers.
(c) The x_i are non-negative integers and $x_8 \leq 2$.

(2) We consider some variations of standard Poker hands. Start with a standard deck of cards (4 suits, 13 values, so 52 cards in total). You will count ways to choose 7 cards.

(a) How many ways can we have two pairs and a triple? i.e., 2 of the cards have the same value, another 2 cards have the same value (but different from the first), and the remaining 3 cards also have the same value.
(b) An "alternator" is a choice of cards such that the values are all distinct, and when they are put in increasing order, the suits alternate (the first, third, fifth, and last card have the same suit S_1, and the second, fourth, and sixth card have the same suit S_2, and $S_1 \neq S_2$). How many are there?
For this problem, A can either be considered the largest or the smallest value (as long as one of the possibilities works).

(3) Let n be a positive integer. Find formulas for $S(n+2, n)$, $S(n+3, n)$, and $S(n+4, n)$ that don't involve Stirling numbers.

(4) Let n be a positive integer such that $n = a_1 + a_2 + a_3$ with all a_i non-negative integers. Generalize the proof of Pascal's identity from lecture to prove the following identity for multinomial coefficients:

$$\binom{n}{a_1, a_2, a_3} = \binom{n-1}{a_1 - 1, a_2, a_3} + \binom{n-1}{a_1, a_2 - 1, a_3} + \binom{n-1}{a_1, a_2, a_3 - 1}.$$

Do this with a bijection rather than using the formula in terms of factorials. The multinomial coefficient is defined to be 0 if any of the bottom arguments are negative.

1. Extra practice problems (don't turn in)

(5) Let $F(n)$ be the number of partitions of $[n]$ such that every block has size ≥ 2. Prove that
$$B(n) = F(n) + F(n+1),$$
where $B(n)$ is the number of partitions of $[n]$.

(6) Continuing with (2): How many ways can we have two triples? i.e., 3 of the cards have the same value, another 3 cards also have the same value, and there is an extra 7th card whose value is different from the others.

(7) Describe a bijection between the set of compositions of n and the set of subsets of $[n-1]$.

(8) Fix an integer $n \geq 2$. Call a composition $(a_1, \dots, a_k)$ of n **doubly even** if the number of a_i which are even is also even (i.e., there could be no even a_i, or 2 of them, or 4, or ...).

Show that the number of doubly even compositions of n is 2^{n-2}.

For example, if $n = 4$, then here are the 4 doubly even compositions of 4:
$$(2,2), \qquad (3,1), \qquad (1,3), \qquad (1,1,1,1).$$
Hint: Given a composition $\alpha = (a_1, \dots, a_k)$, define another composition $\Phi(\alpha)$ by
$$\Phi(\alpha) = \begin{cases} (1, a_1 - 1, a_2, a_3, \dots, a_k) & \text{if } a_1 > 1 \\ (a_2 + 1, a_3, \dots, a_k) & \text{if } a_1 = 1 \end{cases}.$$
(in both cases, we didn't do anything to $a_3, \dots, a_k$). Show that Φ defines a bijection between the set of doubly even compositions of n and the set of compositions of n which are not doubly even.