Math 184, Winter 2022
Homework 4
Due: Friday, Feb. 11 by 11:59PM via Gradescope (late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences.

(1) If $\displaystyle\sum_{n\geq 0} a_n x^n = \frac{2+3x^2-2x^3}{(1-5x)^5}$, find a closed formula for a_n.

(2) Define a sequence by

$$a_0 = 1, \qquad a_1 = 3, \qquad a_n = 8a_{n-1} - 16a_{n-2} + 3^n \qquad \text{for } n \geq 2.$$

   (a) Express $A(x) = \displaystyle\sum_{n\geq 0} a_n x^n$ as a rational function in x.

   (b) Find a closed formula for a_n.

(3) Let $S(n,k)$ be the Stirling number of the second kind. For each $k \geq 1$, define the ordinary generating function

$$\mathbf{S}_k(x) = \sum_{n\geq 0} S(n,k)x^n.$$

   (a) For $k \geq 2$, translate the identity from lecture

$$S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$$

   into an identity involving $\mathbf{S}_k(x)$ and $\mathbf{S}_{k-1}(x)$.

   (b) Use the identity you found in (a) and induction on k to show that for all $k \geq 1$:

$$\mathbf{S}_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}.$$

(4) You want to build a stack of blocks that is n feet high. You have 3 different kinds (unlimited of each): green blocks are 1 foot high, while red and blue blocks are 2 feet high. Blocks of the same color are considered indistinguishable. Let a_n be the number of ways to stack these blocks.

   Find a linear recurrence relation and initial conditions satisfied by a_n.

(5) You are designing a race that takes place over n blocks in a city. It will consist of 3 portions: running, followed by biking, and ending with another running portion. The end of a portion should match up with the end of a block. The first running portion needs to designate 3 blocks to have a first aid tent, and the biking portion needs to designate 4 blocks to have a first aid tent. The second running portion doesn't need anything, but must have positive length. Use generating functions to find a formula for the number of ways to design a race under these conditions.

(6) Let n be a positive integer and let a_n be the number of different ways to pay n dollars using only 1, 2, 5, 10, 20 dollar bills in which at most three 20 dollar bills are used. Express $A(x) = \sum_{n\geq 0} a_n x^n$ as a rational function.