What is this course about?

3 sample problems

① Linear recurrence relations:

Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, ...

can I find formula (non-recursive)?

② Counting # ways to write n pairs of balanced parentheses

$n=2$: ()()   (())

$n=3$: ()()()   (())()   ()(())   ((()))   (()())

general n?

③ How many permutations of n things have no fixed points?

Bijections Given sets X, Y, functions $f: X \to Y$
$g: Y \to X$

we say inverses of each other if: $f \circ g = id_Y$ & $g \circ f = id_X$

i.e., For all $y \in Y$, $f(g(y)) = y$, For all $x \in X$, $g(f(x)) = x$

If so, f and g are bijections.

Prop. If a bijection exists between X, Y then $|X| = |Y|$.

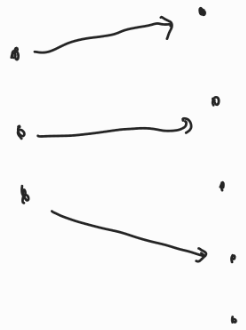
If f is bijection, then it is

- one-to-one (injective), i.e., if $f(x) = f(x')$, then $x = x'$
- onto (surjective), i.e., for all $y \in Y$, there is some x so that $f(x) = y$

f is bijection $\iff$ injective & surjective.

f injective $\Rightarrow$ $|X| \leq |Y|$

f surjective $\Rightarrow$ $|X| \geq |Y|$

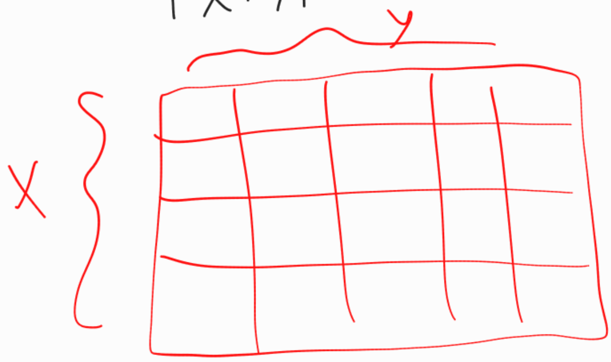


Sum principle: X, Y sets w/ no overlap.

Then $|X \cup Y| = |X| + |Y|$.

Product principle: $X \times Y = \{ (x,y) \mid x \in X, y \in Y \}$

$$|X \times Y| = |X| \cdot |Y|$$



12-fold way: Assigning k balls to n boxes.

subject to some conditions:

Think of assignment as function $f: \{\text{balls}\} \longrightarrow \{\text{boxes}\}$

Conditions on f:

1. f is injective
2. f is surjective
3. no condition

Conditions on balls:

1. all identical
2. considered different

Conditions on boxes:

1. all identical – indistinguishable
2. considered different – distinguishable

balls / boxes	f arbitrary	f injective	f surjective
dist / dist			
indist / dist			
dist / indist		$\begin{cases} 1 & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}$	
indist / indist		$\begin{cases} 1 & \text{if } n \geq k \\ 0 & \text{if } n < k \end{cases}$	

Weak

Induction : prove sequence of statements $P(0), P(1), P(2), \ldots$

Strategy: ① Prove $P(0)$ is true. Base case

② Use $P(n)$ to prove $P(n+1)$ is true. Inductive step

Eg. $P(n)$ is statement "$\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$"

① $P(0)$: "$\sum_{i=0}^{0} i = \frac{0 \cdot (1)}{2} = 0$" ✓

② $$\sum_{i=0}^{n+1} i = \sum_{i=0}^{n} i + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= (n+1)\left(\frac{n}{2} + 1\right) = (n+1)\left(\frac{n+2}{2}\right)$$

$$\frac{n}{2} + 1 = \frac{n}{2} + \frac{2}{2} = \frac{n+2}{2}$$

$$\frac{(n+1)(n+2)}{2}$$

$P(n) \Rightarrow P(n+1)$ ✓

Subsets. S set, T is a subset if every element of T belongs to S.

Note: T could be empty, and also $T = S$ ok.

Thm. If $|S| = n$, there are 2^n subsets of S.

pf. Base case: $n = 0$, $S = \emptyset$, only subset is $T = \emptyset$

$$2^0 = 1 \quad \checkmark$$

Inductive step: Suppose $|S| = n+1 > 0$.

Pick some element $x \in S$. Consider 2 kinds of subsets

I: T contains x

II: T does not contain x

How many of type II?

Define $f : \left\{ \begin{matrix} \text{subsets of } S \\ \text{not containing } x \end{matrix} \right\} \underset{\curvearrowleft g}{\longrightarrow} \left\{ \text{subsets of } S \setminus \{x\} \right\}$

$f(T) = T$, $g(T) = T$, get bijection

$\Rightarrow$ 2^n subsets of type II.

Ex. $S = \{a, b, c\}$, $x = b$

Subsets of S not containing b $= \left\{ \begin{matrix} \emptyset \\ \{a\} \\ \{a, c\} \\ \{c\} \end{matrix} \right\}$

Subsets of $\{a, c\}$ $= \left\{ \begin{matrix} \emptyset \\ \{a\} \\ \{a, c\} \\ \{c\} \end{matrix} \right\}$

How many of type I?

$$\left\{ \begin{matrix} \text{subsets of } S \\ \text{containing } x \end{matrix} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \text{subsets of } S \setminus x \right\}$$

$f(T) = T \setminus x$, $\boxed{g(U) = U \cup \{x\}}$

bijection $\Rightarrow$ 2^n type I subsets.

$\#\text{subsets} = \#\text{type I} + \#\text{type II} = 2^n + 2^n = 2(2^n) = 2^{n+1}$. $\square$

T subset of S not containing x

$$g(f(T)) = g(T \setminus x) = (T \setminus x) \cup \{x\}$$

U subset of $S \setminus x$:

$$f(g(U)) = f(U \cup \{x\}) = U$$

Ex. $$\sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=0}^{n} i^3 = ??$$ deg 4 polynomial in n

$\vdots$

$$\sum_{i=0}^{n} i^d =$$ deg $d+1$ polynomial in n.

$S = \{a, b, c\}$, $x = b$

$\{\cancel{b}\}$ | $\emptyset$
$\{a, \cancel{b}\}$ | $\{a\}$
$\{\cancel{b}, c\}$ | $\{c\}$
$\{a, \cancel{b}, c\}$ | $\{a, c\}$