Let S be any subset of positive integers.

$p_S(n) =$ # partitions of n s.t. all parts belong to S.

<u>Thm</u>. $$\sum_{n \geq 0} p_S(n) x^n = \prod_{i \in S} \frac{1}{1-x^i}.$$

$p_{odd}(n) :=$ # partitions of n s.t. all parts are odd

$p_{dist}(n) :=$ # partitions of n s.t. all parts are distinct.

<u>Thm</u> (Euler). For all n, $p_{odd}(n) = p_{dist}(n)$.

<u>Ex</u>. $n=5$, $p_{odd}(5) = 3$

(5), $(3,1,1)$, $(1,1,1,1,1)$

$p_{dist}(5) = 3$

(5), $(4,1)$, $(3,2)$

<u>Pf</u>. Use previous Thm w/ $S = \{\text{odd positive integers}\}$:

$$\sum_{n \geq 0} p_{odd}(n) x^n = \prod_{i \text{ odd}} \frac{1}{1-x^i} = \frac{1}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots}$$

What about $\sum_{n \geq 0} p_{dist}(n) x^n$? Claim:

$$\sum_{n \geq 0} p_{dist}(n) x^n = \prod_{i \geq 1} (1+x^i) = (1+x)(1+x^2)(1+x^3)(1+x^4)\cdots$$

Why? A term in product is of the form $x^{a_1} x^{a_2} x^{a_3} \cdots x^{a_r}$ where $a_1 < a_2 < a_3 < \cdots < a_r$ finitely many a_i.

$\leftrightarrow$ partition of $n = a_1 + a_2 + a_3 + \cdots + a_r$

namely: $(a_r, a_{r-1}, \ldots, a_1)$ and all a_i are distinct

Ex. $(1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots$

$x^8 = x \cdot x^3 \cdot x^4 \longleftrightarrow (4,3,1)$

Identity: $1-u^2 = (1-u)(1+u)$

$$\frac{1-u^2}{1-u} = 1+u \qquad \text{sub } u \to x^i \qquad \frac{1-x^{2i}}{1-x^i} = 1+x^i$$

$$\sum_{n\geq 0} p_{dist}(n)\, x^n = (1+x)(1+x^2)(1+x^3)(1+x^4)(1+x^5)\cdots$$

$$= \frac{\cancel{1-x^2}}{1-x} \cdot \frac{1-x^4}{\cancel{1-x^2}} \cdot \frac{\cancel{1-x^6}}{1-x^3} \cdot \frac{\cancel{1-x^8}}{\cancel{1-x^4}} \cdot \frac{\cancel{1-x^{10}}}{1-x^5} \cdots$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots}$$

$$= \sum_{n\geq 0} p_{odd}(n)\, x^n$$

$\Rightarrow \quad p_{dist}(n) = p_{odd}(n)$ for all n. $\square$

Catalan numbers

C_n = # ways to write out n pairs of balanced parentheses. $(C_0 = 1)$

Ex. $n=3$.

$()()()$, $(())()$, $((()))$, $(()())$, $()(())$

Define OGF $C(x) = \sum_{n\geq 0} C_n x^n$

Prop. $C(x) = 1 + x\, C(x)^2$

pf. What is recursive structure for balanced parentheses?

$$(\quad w_1 \quad) \quad w_2$$

balanced set of parentheses

matches first one

balanced set of parentheses

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split this up as 2 pieces: (w_1) & w_2

Define $a_n = $ # of balanced pairs of n parentheses of the form (w_1)

$$= \begin{cases} 0 & \text{if } n=0 \\ C_{n-1} & \text{if } n>0 \end{cases}$$

$$x\sum_{n\geq 0} C_{n-1}x^{n-1} = x\sum_{m\geq 0} C_m x^m \qquad m=n-1$$

If $A(x) = \sum_{n\geq 0} a_n x^n = \sum_{n>0} a_n x^n = \sum_{n>0} C_{n-1}x^n = xC(x).$

Conclusion: $\sum_{n\geq 1} C_n x^n = A(x)\,C(x)$ (concatenation interpretation of products of OGF)

$$\Rightarrow \quad C(x) = 1 + \sum_{n\geq 1} C_n x^n = 1 + A(x)C(x) = 1 + xC(x)^2 \quad \square$$

Alternatively: if $n>0$, every balanced set of n pairs is of the form (balanced set of i pairs for some i) balanced set of $n-i-1$ pairs

$$\Rightarrow \quad C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}$$

coeff of x^n in $xC(x)^2$

$$xC(x)^2 = x\Big(\sum_{n\geq 0} C_n x^n\Big)^2 = \sum_{m\geq 0}\Big(\sum_{i=0}^{m} C_i C_{m-i}\Big)x^{m+1}$$

coeff of x^n: set $m = n-1$: $\sum_{i=0}^{n-1} C_i C_{n-1-i}$

$\Rightarrow$ $$x\,C(x)^2 - C(x) + 1 = 0$$

i.e., $C(x)$ is a root to quadratic polynomial

$$x t^2 - t + 1$$

$\Rightarrow$ $\exists$ choice of sign s.t.

$$C(x) = \frac{1 \pm (1-4x)^{1/2}}{2x}$$

numerator must be divisible by x

constant term of $(1-4x)^{1/2}$ is just $\binom{1/2}{0} = 1$

$-$ is correct:

$$C(x) = \frac{1 - (1-4x)^{1/2}}{2x}$$

Thm. $C_n = \frac{1}{n+1}\binom{2n}{n}$.

Pf. Use (general) binomial thm.

$$(1-4x)^{1/2} = \sum_{n\geq 0} \binom{1/2}{n}(-4x)^n = \sum_{n \geq 0} (-4)^n \binom{1/2}{n} x^n$$

Assume $n > 0$:

$$(-1)^n 4^n \binom{1/2}{n} = (-1)^n 4^n \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\cdots\left(-\frac{(2n-3)}{2}\right)}{n!}$$

$$= -\frac{4^n}{2^n}\frac{1\cdot 1\cdot 3\cdots(2n-3)}{n!} = -2^n\frac{(2n-3)!!}{n!} \quad (*)$$

Note: $(2n-3)!!\,(2n-2)!! = (2n-2)!$

Also, $(2n-2)!! = (2n-2)(2n-4)\cdots 2 = 2^{n-1}\cdot(n-1)!$

$$(*) = -2^n\frac{(2n-3)!!\,(2n-2)!!}{(2n-2)!!\;n!} = -2^n\frac{(2n-2)!}{2^{n-1}(n-1)!\,n!} = -\frac{2}{n}\binom{2n-2}{n-1}$$

$$C(x) = \frac{1-(1-4x)^{1/2}}{2x} = \frac{-\sum_{n\geq 1}\binom{1/2}{n}(-4x)^n}{2x}$$

$$= -\frac{1}{2}\sum_{n\geq 1}(-4)^n\binom{1/2}{n}x^{n-1} = -\frac{1}{2}\sum_{n\geq 1}\frac{-2}{n}\binom{2n-2}{n-1}x^{n-1}$$

$m = n-1$

$$-\frac{1}{\cancel{2}}\sum_{m\geq 0}\frac{-\cancel{2}}{m+1}\binom{2m}{m}x^m$$

$$\Rightarrow \quad C(x) = \sum_{m\geq 0}\frac{1}{m+1}\binom{2m}{m}x^m$$

$$\Rightarrow \quad C_n = \frac{1}{n+1}\binom{2n}{n} \text{ for all } n\geq 0. \qquad \square$$

Ex. $n=3$: $\frac{1}{4}\binom{6}{3} = \frac{1}{4}\cdot 20 = 5$ ✓

Other interpretations:

① Given binary operation $*$, inputs 2 things, outputs 1

$a * b$

C'_n = # ways to apply $*$ to $n+1$ inputs:

$n=3$:

$a*(b*(c*d))$ $\quad$ $a*((b*c)*d)$ $\quad$ $(a*b)*(c*d)$

$((a*b)*c)*d$ $\quad$ $(a*(b*c))*d$

$$\boxed{C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i} \quad \text{for } n \geq 1.}$$

Consider last time $*$ is used.

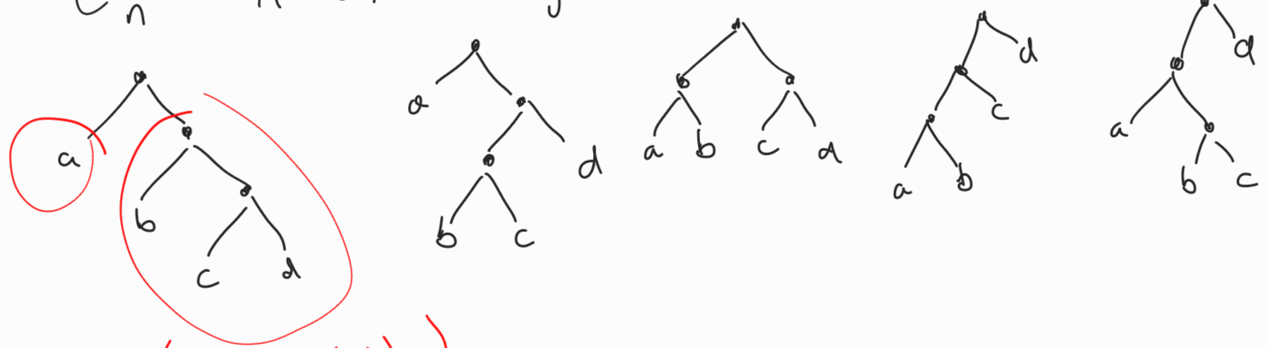
Left input has i symbols, $1 \leq i \leq n$ $\rightarrow$ C'_{i-1} possibilities

Right input has $n+1-i$ symbols $\rightarrow$ C'_{n-i} possibilities

$$\Rightarrow C'_n = \sum_{i=1}^{n} C'_{i-1} C'_{n-i} \qquad j = i-1$$

$$= \sum_{j=0}^{n} C'_j C'_{n-1-j}$$

② C''_n = # rooted binary trees w/ $n+1$ leaves.



$a*(b*(c*d))$