last time: sequence $(a_n) \longrightarrow$ EGF $A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$

product $A(x)B(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n$

where c_n = # ways to choose subset $S \subseteq [n]$
and put structure α (counted by a_i) on S
structure β (counted by b_j) on $[n] \setminus S$.

Ex. Consider $a_n = 1$ for $n > 0$
$a_0 = 0$

$$A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n = e^x - 1$$

$$A(x)A(x) = \sum_{n\geq 0} \frac{c_n}{n!} x^n \quad \text{where}$$

c_n = # ways to pick $S \subseteq [n]$ s.t.
$S \neq \emptyset$ & $[n]\setminus S \neq \emptyset$
= ordered set partitions of $[n]$ into 2 blocks
$= 2!\, S(n,2)$

More generally,

$$A(x)^k = \sum_{n\geq 0} \frac{c_n}{n!} x^n \quad \text{where}$$

c_n = # ways to pick subsets $S_1, \ldots S_k \subseteq [n]$ s.t.
each $S_i \neq \emptyset$, $S_1 \cup \ldots \cup S_k = [n]$
S_i's don't overlap.
= # ordered set partitions of $[n]$ into k blocks
$= k!\, S(n,k)$

$$\Rightarrow \sum_{n\geq 0} k!\, S(n,k) \frac{x^n}{n!} = A(x)^k = (e^x - 1)^k$$

$$\sum_{n\geq 0} \frac{S(n,k)}{n!} x^n = \frac{(e^x-1)^k}{k!}$$

Bell numbers: $B(n)$: # set partitions of $[n]$

$$= \sum_{k\geq 0} S(n,k)$$

$$\sum_{n\geq 0} B(n) \frac{x^n}{n!} = \sum_{n\geq 0} \sum_{k\geq 0} S(n,k) \frac{x^n}{n!} = \sum_{k\geq 0} \sum_{n\geq 0} S(n,k) \frac{x^n}{n!}$$

$$= \sum_{k\geq 0} \frac{(e^x-1)^k}{k!} = e^{e^x - 1} = \exp(e^x - 1).$$

Composition of EGF:

Let a_n be sequence s.t. $a_0 = 0$

Think of a_n = # structures of type α on set of size n.

Let h_n = # ways to pick set partition of $[n]$ and put structure α on each block.

Thm (Exponential formula)

Let $A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n$, $H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n$.

Then $H(x) = \exp(A(x)) = e^{A(x)}$.

Pf. Write $A(x)^k = \sum_{n\geq 0} \frac{c_n}{n!} x^n$

Then c_n = # ways to pick k subsets $S_1, \ldots S_k$ s.t. $S_1 \cup \ldots \cup S_k = [n]$, S_i's don't overlap, we put structure α on each S_i.

Since $a_0 = 0$, these sets must be nonempty.

$\Rightarrow \frac{c_n}{k!}$ = # ways to pick set partition of $[n]$ into k blocks and put structure α on each block

$$\Rightarrow \sum_{k\geq 0} \frac{A(x)^k}{k!} = \sum_{n\geq 0} \frac{h_n}{n!} x^n$$

$$\parallel$$

$$\exp(A(x))$$

$\square$

Prop. If $H(x) = e^{A(x)}$, then

$$H'(x) = H(x) A'(x).$$

Ex. A bijective function $f: S \to S$ is an involution if $f \circ f = id$ (equiv, $f^{-1} = f$)

How many involutions on a set of size n? Call this h_n.

Another perspective: involution is same as writing S as disjoint union of subsets all of size 1 or 2.

Ex. $S = \{1,2,3,4,5,6\}$

$f(1)=1 \quad f(4)=6$
$f(2)=3 \quad f(5)=5$
$f(3)=2 \quad f(6)=4$

$(1) \quad (2\ 3) \quad (4\ 6) \quad (5)$

$$a_n = \begin{cases} 1 & \text{if } n=1,2 \\ 0 & \text{else} \end{cases} = \text{counts structure which selects for size 1 or 2 sets.}$$

$$A(x) = \sum_{n\geq 0} \frac{a_n}{n!} x^n = x + \frac{x^2}{2}, \quad \text{Define } H(x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n$$

Exp. formula: $H(x) = e^{A(x)} = e^{x+\frac{x^2}{2}}$

Take derivative: $H'(x) = H(x)A'(x) = H(x)(1+x)$

$$\sum_{n\geq 0} \frac{n h_n}{n!} x^{n-1} \qquad \left(\sum_{n\geq 0} \frac{h_n}{n!} x^n\right)(1+x) = \sum_{n\geq 0} \frac{h_n}{n!} x^n + \sum_{n\geq 0} \frac{h_n}{n!} x^{n+1}$$

Take coeff. of x^r on both sides (assume $r\geq 1$):

$$\frac{(r+1)h_{r+1}}{(r+1)!} = \frac{h_r}{r!} + \frac{h_{r-1}}{(r-1)!}$$

multiply by $r!$: $\boxed{h_{r+1} = h_r + r \cdot h_{r-1} \quad \text{for } r\geq 1}$

initial values: $h_0 = 1$
$h_1 = 1$

Interpreting $h_{r+1} = h_r + r h_{r-1}$:

Two types of involutions f of $[r+1]$:

Type I: $f(r+1) = r+1$. $\# =$ involutions on $[r] = h_r$

Type II: $f(r+1) \neq r+1$. r choices for $f(r+1)$

$+$ choose involution on $[r] \setminus \{\text{choice}\}$

$= r \cdot h_{r-1}$.

Ex. Let $h_n =$ # ways to break n people into nonempty groups and cyclically order each group

$$H(x) = \sum_{n \geq 0} \frac{h_n}{n!} x^n.$$

Let $a_n =$ # ways to cyclically order n things if $n > 0$.

(0 if $n=0$)

$$= \begin{cases} (n-1)! & \text{if } n > 0 \\ 0 & \text{if } n = 0 \end{cases}.$$

Define $A(x) = \sum_{n \geq 1} (n-1)! \frac{x^n}{n!} = \sum_{n \geq 1} \frac{x^n}{n}$

$$A'(x) = \sum_{n \geq 1} \frac{n x^{n-1}}{n} = \sum_{n \geq 1} x^{n-1} = \frac{1}{1-x}$$

Exp. formula: $H(x) = \exp(A(x))$.

Take derivative: $H'(x) = H(x) A'(x)$

$$(1-x) H'(x) = H(x)$$

$$(1-x) \sum_{n \geq 0} \frac{n h_n}{n!} x^{n-1} = \sum_{n \geq 0} \frac{h_n}{n!} x^n$$

$$\sum_{n \geq 1} \frac{h_n}{(n-1)!} x^{n-1} - \sum_{n \geq 1} \frac{h_n}{(n-1)!} x^n$$

"

Take coeff. of x^r:

$$\frac{h_{r+1}}{r!} - \frac{h_r}{(r-1)!} = \frac{h_r}{r!}$$

multiply by $r!$: $h_{r+1} - r h_r = h_r$

$h_{r+1} = (r+1) h_r \quad r \geq 1, \quad h_0 = 1$

$\Rightarrow h_r = r!$

Cyclic ordering:

claim we have bijection:

$$\left\{\begin{matrix}\text{cyclic orderings} \\ \text{on } [n]\end{matrix}\right\} \times [n] \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{matrix}\text{permutations} \\ \text{of } [n]\end{matrix}\right\}$$

σ

$\downarrow$

sequence $(\sigma(1), \sigma(2), \ldots, \sigma(n))$

$$g(\sigma) = \left(\begin{matrix} & \sigma(1) & \\ \sigma(n) & & \sigma(2) \\ & & \sigma(3) \\ & \ldots & \end{matrix}, \ \sigma(1) \right)$$

---

general compositions of EGF:

a_n = number of ways to put structure α on set of size n.

Assume $a_0 = 0$.

b_k = number of ways to put structure β on set of size k.

h_n = number of ways to pick set partition of $[n]$,

put structure α on each block +

put structure β on set of blocks

Define $A(x) = \sum_{n \geq 0} \frac{a_n}{n!} x^n$, $B(x) = \sum_{n \geq 0} \frac{b_n}{n!} x^n$, $H(x) = \sum_{n \geq 0} \frac{h_n}{n!} x^n$.

Thm (Composition formula, exponential version)

$$H(x) = B(A(x)).$$

Cayley's formula for # labeled trees.

Def. S = set, A labeled (simple) graph on S is a collection of two-element subsets of S.

S = "vertices"

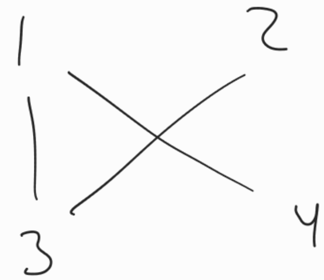
the two-element subsets = "edges".

Visualize: Draw elements of S as points

Draw lines joining i, j if $\{i, j\}$ is an edge

$S = \{1, 2, 3, 4\}$.

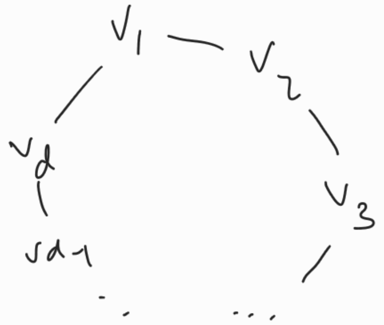
edges: $\{1,3\}$, $\{1,4\}$, $\{2,3\}$



labeled graphs on set of size n

$$= 2^{\binom{n}{2}}$$

Def. A *cycle* in a labeled graph is a sequence $v_1, v_2, \ldots, v_d$ $(d \geq 3)$ $\vee$ s.t. all v_i distinct, $\{v_i, v_{i+1}\}$ is an edge for $i = 1, \ldots d-1$
$\{v_d, v_1\}$ is an edge.



A *labeled forest* is a labeled graph w/ no cycles.

A *labeled tree* is a labeled forest which is connected (i.e., for any two vertices, there is a way to get from one to the other by following edges)

Let t_n = # labeled trees w/ n vertices.

Thm (Cayley). $t_n = n^{n-2}$.

Ex. $n = 1$. 1 $\qquad t_1 = 1^{-1} = 1$

$n = 2$ 1 — 2 $\qquad t_2 = 2^0 = 1$

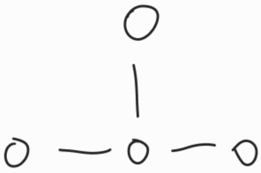
$\qquad t_3 = 3^1 = 3$

$n = 3$: 1 — 3 — 2 $= \{1,3\}, \{2,3\}$

2 — 1 — 3

1 — 2 — 3

$\underline{n=4}$.



$t_4 = 4^2 = 16$

4

$4!/2 = 12$

$\parallel$