labeled graphs: nodes $1, \ldots, n$ w/ edges connecting them

$\implies 2^{\binom{n}{2}}$ many

labeled forests: graphs w/ no cycles.

labeled trees: connected forests. t_n = # labeled trees.

Cayley: $t_n = n^{n-2}$ for $n \geq 1$.

Ex: $n=3$ 

1 — 2 — 3

2 — 1 — 3

2 — 3 — 1

$n=4$  

4 $\qquad$ $4!/2 = 12$

Rooted labeled tree = pair (T, i) T = labeled tree on n vertices

$1 \leq i \leq n$

rooted labeled trees = $n t_n$

Planted labeled forest = labeled forest + choice of root for each connected component

= disjoint union of labeled trees (union of labels is $[n]$)

f_n := # planted labeled forests w/ n vertices. $(f_0 = 1)$

$$F(x) := \sum_{n \geq 0} f_n \frac{x^n}{n!}, \quad R(x) := \sum_{n \geq 0} (n t_n) \frac{x^n}{n!}$$

Identity 1: $F(x) = e^{R(x)}$

Pf: To build planted labeled forest on n vertices:

① Pick set partition $X_1, \ldots, X_k$ of $[n]$

② Put structure of rooted labeled tree on each X_i.

This gives every planted labeled forest exactly once.

Exponential formula $\Longrightarrow$ $F(x) = e^{R(x)}$. $\square$

Identity 2. $R(x) = x F(x)$

Pf. Claim: For $n \geq 1$, $t_n = f_{n-1}$. Use bijection.

$$\left\{ \begin{matrix} \text{labeled trees} \\ \text{on } [n] \end{matrix} \right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{ \begin{matrix} \text{planted labeled} \\ \text{forests on } [n-1] \end{matrix} \right\}$$

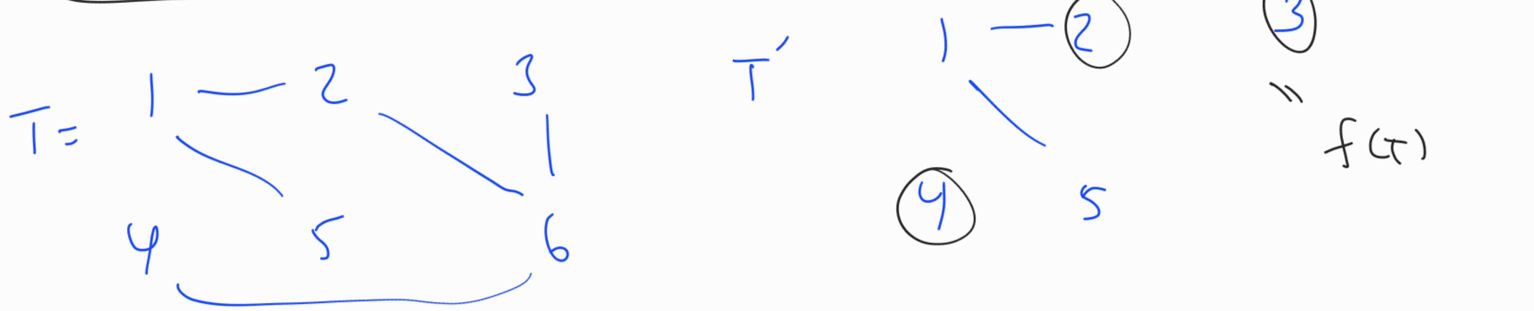
Let T be labeled tree w/ vertices $1, \ldots, n$.

Delete vertex labeled n + all edges containing n.

Result T' has no cycles, hence is a labeled forest on $[n-1]$

For each connected component of T', there is a unique vertex that was connected to n by an edge in T. Let that be the root of the component.

Now I have planted labeled forest, call it $f(T)$.



Let u be planted labeled forest on $[n-1]$.

Add vertex n. For each root in each component of u, add edge between n and that root.

Get labeled tree on $[n]$, call it $g(u)$.

$\Rightarrow$ f, g inverses, get bijection that proves $t_n = f_{n-1}$.

$$R(x) = \sum_{n\geq 1} (n t_n) \frac{x^n}{n!} = \sum_{n\geq 1} n f_{n-1} \frac{x^n}{n!} = x \sum_{n\geq 1} f_{n-1} \frac{x^{n-1}}{(n-1)!} = x F(x).$$

$\square$

$$R(x) = x F(x) = x e^{R(x)}$$

$$\Rightarrow \boxed{R(x) = x e^{R(x)}}$$

Try to solve for coefficients of $R(x)$.

Let $r_n = \begin{cases} \frac{n t_n}{n!} & \text{for } n>0 \\ 0 & \text{for } n=0 \end{cases}$, so $R(x) = \sum_{n\geq 0} r_n x^n$

$$r_1 = [x^1] R(x) = [x^1]\left(x e^{R(x)}\right) = [x^0] e^{R(x)}$$

Note: constant term of $R(x)$ is 0

so $[x^0] e^{R(x)} = [x^0]\left(1 + \cancel{R(x)} + \cancel{\frac{R(x)^2}{2!}} + \dots\right) = 1$

$$r_2 = [x^2]R(x) = [x^2]\left(xe^{R(x)}\right) = [x^1]\, e^{R(x)}$$

$$= [x^1]\left(\cancel{1} + R(x) + \cancel{\frac{R(x)^2}{2!}} + \cancel{\frac{R(x)^3}{3!}} + \dots\right) = r_1 = 1$$

$$r_3 = [x^3]\, R(x) = [x^3]\left(x e^{R(x)}\right) = [x^2]\, e^{R(x)}$$

$$= [x^2]\left(\cancel{1} + R(x) + \frac{R(x)^2}{2!} + \cancel{\frac{R(x)^3}{3!}} + \cancel{\frac{R(x)^4}{4!}} + \dots\right)$$

$$= r_2 + \frac{1}{2!}\left(r_0 r_2 + r_1^2 + r_2 r_0\right) = 1 + \frac{1}{2}(0+1+0)$$

$$= 3/2.$$

Lagrange inversion formula: Let $G(x)$ be FPS w/ nonzero constant term. Then there is a unique FPS $A(x)$ s.t.

$$A(x) = x\, G(A(x)).$$

Furthermore, $[x^0]A(x) = 0$ and for $n > 0$,

$$[x^n]\, A(x) = \frac{1}{n}[x^{n-1}]\left(G(x)^n\right).$$

Conclusion of Cayley's formula:

Use Lagrange inversion w/ $G(x) = e^x$.

For $n > 0$,

$$[x^n]\, R(x) = \frac{1}{n}[x^{n-1}]\, e^{nx} = \frac{1}{n}[x^{n-1}] \sum_{k\geq 0} \frac{n^k x^k}{k!}$$

Set $k = n-1$:

$$= \frac{1}{n}\frac{n^{n-1}}{(n-1)!}$$

By definition, $[x^n] = \frac{n t_n}{n!}$

$$\Rightarrow \frac{n t_n}{n!} = \frac{n^{n-1}}{n!} \Rightarrow t_n = n^{n-2} \qquad \square$$

(other method: Prüfer encoding)

Catalan numbers (again)

Catalan numbers.

Recall: $C(x) = \sum_{n \geq 0} C_n x^n$

We showed: $C(x) = 1 + x C(x)^2$

Define $A(x) = C(x) - 1.$

$$\Rightarrow A(x) \cancel{+1} = \cancel{1} + x(A(x)+1)^2$$

$$\Rightarrow A(x) = x(A(x)+1)^2$$

Can apply Lagrange inversion w/ $G(x) = (x+1)^2$

For $n > 0$,

$$[x^n] A(x) = \frac{1}{n}[x^{n-1}](x+1)^{2n} = \frac{1}{n}[x^{n-1}] \sum_{k=0}^{2n} \binom{2n}{k} x^k$$

$$= \frac{1}{n}\binom{2n}{n-1}$$

Note: $\frac{1}{n}\binom{2n}{n-1} = \frac{(2n)!}{n \cdot (n-1)! (n+1)!} = \frac{1}{n+1} \frac{(2n)!}{n! n!} = \frac{1}{n+1}\binom{2n}{n}.$

Also, for $n > 0$, $[x^n] A(x) = [x^n] C(x).$

Generalized Catalan:

Recall: $C_n = \#$ rooted binary trees w/ $n+1$ leaves.

$= \#$ rooted binary trees w/ n internal vertices

Let $k \geq 2$ integer.

Consider $\#$ rooted k-ary trees w/ n internal vertices

Let's call this α_n

$$\Rightarrow \quad \alpha_n = \sum_{\substack{(i_1, i_2, \dots, i_k) \\ i_1 + \dots + i_k = n-1}} \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_k} \qquad (*) \text{ for } n > 0$$

$\Rightarrow$ Define $B(x) = \sum_{n \geq 0} \alpha_n x^n$.

$(*)$ translates to $\boxed{B(x) = 1 + x B(x)^k}$

Define $A(x) = B(x) - 1$:

$$A(x) = x(A(x)+1)^k$$

Use Lagrange w/ $G(x) = (x+1)^k$.

$$[x^n] A(x) = \frac{1}{n}[x^{n-1}](x+1)^{nk} = \frac{1}{n}[x^{n-1}] \sum_{i=0}^{nk} \binom{nk}{i} x^i$$

$$= \frac{1}{n}\binom{nk}{n-1}$$

$$B(x) = \sum_{n\geq 0} b_n x^n.$$

What is coeff of x^n in $B(x)^k$?

Claim: $\displaystyle\sum_{\substack{(i_1,\dots,i_k)\\ i_1+\dots+i_k=n}} b_{i_1}\cdots b_{i_k}$

Prove by induction.

When $k=2$, $\displaystyle\sum_{i=0}^{n} b_i b_{n-i} = \sum_{\substack{(i,j)\\ i+j=n}} b_i b_j$

For $k>2$, $B(x)^k = B(x)^{k-1}B(x)$

$$[x^n]B(x)^k = [x^n]\,B(x)^{k-1}B(x)$$
$$= \sum_{j=0}^{n} [x^j]B(x)^{k-1}\, b_{n-j}$$
$$= \boxed{\sum_{j=0}^{n} \sum_{\substack{(i_1,\dots,i_{k-1})\\ i_1+\dots+i_{k-1}=j}} b_{i_1}\cdots b_{i_{k-1}}\, b_{n-j}}$$

Every weak composition of n into k parts is of the form $(i_1,\dots,i_{k-1},n-j)$ where $i_1+\dots+i_{k-1}=j$ for some $0\leq j\leq n$.