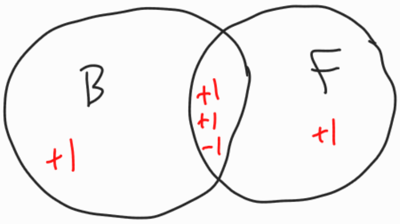
Inclusion-exclusion.

Example. 2 sets.

14 basketball players in room
+ 10 football players in room $\Rightarrow$ How many athletes?



$$|B \cup F| = |B| + |F| - |B \cap F|$$

$$|B \cup F| = ?$$

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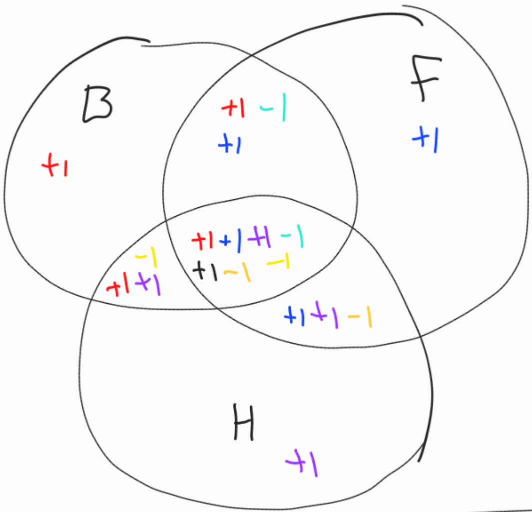
3 sets. + 8 hockey players in room $\Rightarrow$ How many athletes?



$$|B \cup F \cup H| = |B| + |F| + |H|$$

$$- |B \cap F| - |B \cap H| - |H \cap F|$$

$$+ |B \cap F \cap H|$$

Thm (Inclusion-exclusion) Let $A_1, \ldots, A_n$ be (finite) sets.

$$|A_1 \cup \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}|$$

$$|A_1 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n| \quad (j=1)$$

$$- |A_1 \cap A_2| - |A_1 \cap A_3| - \cdots - |A_1 \cap A_n| \quad (j=2)$$

$$- |A_2 \cap A_3| - |A_2 \cap A_4| - \cdots - |A_2 \cap A_n|$$

$$- \quad \cdots \quad - |A_{n-1} \cap A_n|$$

$$+ |A_1 \cap A_2 \cap A_3| + \cdots + |A_x \cap A_y \cap A_z| + \cdots + |A_{n-2} \cap A_{n-1} \cap A_n| \qquad (j=3)$$

$$\vdots$$

$$+ (-1)^{n-1} |A_1 \cap \cdots \cap A_n|. \qquad (j=n)$$

<u>Pf</u>. Pick $x \in A_1 \cup \cdots \cup A_n$. Goal: Show x is counted exactly once on RHS.

Let $s_1, \ldots, s_k$ be all indices s.t. $x \in A_{s_i}$

$$S := \{s_1, \ldots, s_k\}.$$

Then $x \in A_{i_1} \cap \cdots \cap A_{i_j} \iff i_1, \ldots, i_j \in S$.

$\Rightarrow$ Contribution in the sum for x is

$$\sum_{\substack{T \subseteq S \\ T \neq \emptyset}} (-1)^{|T|-1} = \sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} = 1$$

From binomial Thm,
$$\sum_{i=0}^{k} (-1)^{i-1} \binom{k}{i} = 0 \quad \text{for } k > 0$$
$$\|$$
$$\sum_{i=1}^{k} (-1)^{i-1} \binom{k}{i} + (-1)$$

<u>Derangements:</u>

n people w/ hats. Exchange randomly.

How many ways so that no one has their own hat?

What is probability no one has their own hat?

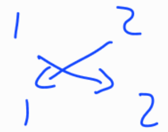
Mathematical formulation:

A <u>derangement</u> of S is a bijection $f: S \to S$ s.t. $\forall x \in S$, $f(x) \neq x$.

How many derangements of a set of size n?

<u>EX</u>. $n=1$. 0

$n=2$



1

$n=3$

2

<u>Thm</u>. The number of derangements of a set of size n is

$$\sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$$

<u>Pf</u>. We will count non-derangements, i.e. bijections f s.t. $f(x)=x$ for some x. Let set be $[n]$.

f is a derangement if: $f(1)=1$ OR
$f(2)=2$ OR
$f(3)=3$ OR
$\vdots$
$f(n)=n$

Define A_i to be $\{f \mid f \text{ permutation } \& \ f(i)=i\}$.

Non-derangements $= A_1 \cup \dots \cup A_n$.

How big is A_1? If $f \in A_1$, then $f(1)=1$ but otherwise no restrictions.

$A_1 \longleftrightarrow$ permutations of $\{2, \dots, n\}$

$\Longrightarrow |A_1| = (n-1)!$

Also, $|A_i| = (n-1)!$

How about $|A_1 \cap A_2|$?

$f \in A_1 \cap A_2$ if $f(1)=1$, $f(2)=2$
and otherwise no restrictions.

$A_1 \cap A_2 \longleftrightarrow$ permutations of $\{3, \dots, n\}$.

$\Longrightarrow |A_1 \cap A_2| = (n-2)!$

If $i \neq j$, $|A_i \cap A_j| = (n-2)!$

In general, for any indices $i_1 < i_2 < \dots < i_j$:

$A_{i_1} \cap \dots \cap A_{i_j} \longleftrightarrow$ permutations of $[n] \setminus \{i_1, \dots, i_j\}$

$\Longrightarrow |A_{i_1} \cap \dots \cap A_{i_j}| = (n-j)!$

I-E: $$|A_1 \cup \dots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < \dots < i_j \le n} |A_{i_1} \cap \dots \cap A_{i_j}|$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < \dots < i_j \le n} (n-j)!$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \binom{n}{j} (n-j)!$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \frac{n!}{j!\,\cancel{(n-j)!}} \cancel{(n-j)!} = \sum_{j=1}^{n} (-1)^{j-1} \frac{n!}{j!}$$

$$\#\text{ derangements} = n! - \sum_{j=1}^{n} (-1)^{j-1} \frac{n!}{j!} = \boxed{\sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!}} \quad \square$$

Problem w/ alternating sums: how to estimate size?

Extreme example: for fixed i, $\binom{n}{i} \to \infty$ as $n \to \infty$

but $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$

Observation. For any real number r,

$$e^r = \sum_{i=0}^{\infty} \frac{r^i}{i!}$$

when $r=-1$, first $n+1$ terms gives

$\frac{\#\text{ derangements}}{n!}$ = probability that randomly chosen permutation is a derangement.

As $n \to \infty$, this tells us that probability $\to 1/e \approx 0.368$

Approximate using Taylor's remainder Thm.

$$e^{-1} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = \underbrace{\sum_{i=0}^{n} \frac{(-1)^i}{i!}}_{\text{probability of being a derangement}} + \underbrace{\sum_{i=n+1}^{\infty} \frac{(-1)^i}{i!}}_{= R_n}$$

bound this?

Lagrange's version of Taylor: $\left| \sum_{i=n+1}^{\infty} \frac{(-1)^i}{i!} \right| \leq \frac{1}{(n+1)!}$

$$\frac{n!}{e} = \#\text{ derangements} + n! R_n$$

$|n! R_n| \leq \frac{1}{n+1}$

derangements $\in \left[\frac{n!}{e} - \frac{1}{n+1}, \frac{n!}{e} + \frac{1}{n+1}\right]$

$\Rightarrow$ # derangements $= \text{round}\left(\frac{n!}{e}\right)$

Approach 2 to derangements using EGF:

Observation: every permutation of $[n]$ is built as follows:

Pick subset $S \subseteq [n]$ and set $f(i) = i$ for all $i \in S$

Pick derangement of $[n] \setminus S$

Two structures: Structure 1: put identity function on set.

1 way to do this

$\rightsquigarrow$ $\text{EGF} = \sum_{n \geq 0} \frac{x^n}{n!} = e^x$

Structure 2: pick a derangement on a set.

$d_n =$ # derangements of $[n]$

$\text{EGF} = \sum_{n \geq 0} \frac{d_n x^n}{n!} =: D(x)$.

$\Rightarrow$ $e^x D(x) =$ EGF for picking permutation

$= \sum_{n \geq 0} \frac{n!}{n!} x^n = \frac{1}{1-x}$.

$\Rightarrow$ $D(x) = e^{-x}\left(\frac{1}{1-x}\right)$.

$\frac{d_n}{n!} = [x^n] D(x) = [x^n]\, e^{-x}\left(\frac{1}{1-x}\right)$

$= \sum_{i=0}^{n} [x^i] e^{-x} [x^{n-i}] \left(\frac{1}{1-x}\right)$

$$= \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

$$\Rightarrow d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}$$

Stirling numbers (revisited)

Thm. For all $n \geq k > 0$,

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n = \sum_{i=0}^{k} (-1)^i \frac{(k-i)^n}{i!(k-i)!}$$

Pf. We will count ordered set partitions of $[n]$ into k blocks.

same as # surjective functions $[n] \to [k]$

We will count non-surjective functions

f fails to be surjective if

1 is not in image

OR 2 is not in image

OR...

n is not in image

Define $A_i = \{ f : [n] \to [k] \mid i$ is not in image of $f \}$.

$|A_1|$? $A_1 \leftrightarrow$ functions $[n] \to \{2, \ldots, k\}$

$\Rightarrow |A_1| = (k-1)^n$

$\Rightarrow |A_i| = (k-1)^n$ for any i.

For any indices $i_1 < i_2 < \cdots < i_j$,

$A_{i_1} \cap \cdots \cap A_{i_j} \leftrightarrow$ functions $[n] \to [k] \setminus \{i_1, \ldots, i_j\}$

$\Rightarrow |A_{i_1} \cap \cdots \cap A_{i_j}| = (k-j)^n$

$$|A_1 \cup \dots \cup A_k| = \sum_{j=1}^{k} (-1)^{j-1} \sum_{1 \le i_1 < \dots < i_j \le k} |A_{i_1} \cap \dots \cap A_{i_j}|$$

$$= \sum_{j=1}^{k} (-1)^{j-1} \sum_{1 \le i_1 < \dots < i_j \le k} (k-j)^n$$

$$= \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (k-j)^n$$

$$\Rightarrow \quad S(n,k) = \frac{1}{k!} \left(k^n + \sum_{j=1}^{k} (-1)^j \binom{k}{j} (k-j)^n \right)$$

$$= \frac{1}{k!} \sum_{j=0}^{n} (-1)^j \binom{k}{j} (k-j)^n . \qquad \square$$